ONERA — THE FRENCH AEROSPACE LAB — return on innovation

Inria — informatics mathematics

Atom Selection in Continuous Dictionaries : Reconciling Polar and SVD Approximations

Frédéric Champagnat and Cédric Herzet

Contact : frederic.champagnat@onera.fr

Problem Statement

A generic problem : Atom selection in a continuous dictionary

$$\boldsymbol{a}_{\max} \in \underset{\boldsymbol{a}\in\mathcal{A}}{\arg\max}\langle \boldsymbol{a}, \boldsymbol{r}\rangle$$

- $\mathcal{A}$ dictionary of **infinitely uncountable** atoms **a**(θ) in some Hilbert space $\mathcal{H}$
  θ indexed on an interval Θ of R^d
- ***r*** is some iteration dependent « residual » vector in $\mathcal{H}$
- Such a problem typically occurs in critical steps of BLASSO or OMP on continuous dictionaries

- **Typical applications : particle imaging (PIV, PALM-STORM)**

A common assumption

"*Some efficient scheme to find a_{max} exists*"

Proposed approach

- **A general framework for continuous dictionary approximation**
  Piecewise linear approximation
- **Efficient atom selection through *Polar approximation***
  Accurate closed-form expression
- **||.||-optimal linear approximation framework (i.e. SVD)**
  Raised-cosine dictionary kernels yield Polar approximation

Dictionary Approximation

Piecewise Linear Approximation

$$\hat{\mathcal{A}} = \cup_{\ell=1}^{L}\hat{\mathcal{A}}_\ell \qquad \hat{\mathcal{A}}_\ell = \left\{\sum_{k=1}^{K} c_k \boldsymbol{v}_k : \mathbf{c}\in\mathcal{C}\right\} \qquad \boldsymbol{v}_k \in \mathcal{H} \text{ and } \mathcal{C} \subseteq \mathbb{R}^K$$

- $\hat{\mathcal{A}}_\ell$ **is an approximation of some subdictionary** $\mathcal{A}_\ell = \{\boldsymbol{a}(\theta) : \theta \in \Theta_\ell\}$.

$$\max_{\boldsymbol{a}\in\hat{\mathcal{A}}}\langle \boldsymbol{a},\boldsymbol{r}\rangle = \max_{\ell}\max_{\boldsymbol{a}\in\hat{\mathcal{A}}_\ell}\langle \boldsymbol{a},\boldsymbol{r}\rangle$$

- **Accurate atom selection promotes $K \geq 3$ (1D case)**
  - piecewise constant (K =1) : no discrimination within Θ_l
  - Taylor 1st order (K = 2) : optimal argument at the boundary of Θ_l
- ***Polar approximation*** $\hat{\mathcal{A}}_\ell = \left\{\boldsymbol{v}_1 + \cos(\omega\theta)\boldsymbol{v}_2 + \sin(\omega\theta)\boldsymbol{v}_3 : \tau \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]\right\}$

$$\arg\max_{\tau\in\left[-\frac{\Delta}{2},\frac{\Delta}{2}\right]}\langle \boldsymbol{v}_1,\boldsymbol{r}\rangle + \cos(\omega\tau)\langle \boldsymbol{v}_2,\boldsymbol{r}\rangle + \sin(\omega\tau)\langle \boldsymbol{v}_3,\boldsymbol{r}\rangle$$

Admits a closed-form expression

||.||-optimal Approximation (SVD)

- **Determine optimal** $V = \mathrm{span}(\{\boldsymbol{v}_\ell\}_{k=1}^{K})$ **through**

$$V_{\mathrm{opt}}^{K} = \underset{V:\dim(V)=K}{\arg\min}\int_{\Theta_\ell}\|\boldsymbol{a}(\theta) - P_V(\boldsymbol{a}(\theta))\|^2 d\theta. \qquad P_V(\boldsymbol{a}) \triangleq \underset{\tilde{\boldsymbol{a}}\in V}{\arg\min}\|\boldsymbol{a}-\tilde{\boldsymbol{a}}\|$$

- **Solution : spectral decomposition of kernel operator**

$$\kappa(\theta,\theta') \triangleq \langle \boldsymbol{a}(\theta), \boldsymbol{a}(\theta')\rangle \Rightarrow R: L_2(\Theta_\ell)\to L_2(\Theta_\ell),\ u(\theta)\mapsto \int_{\Theta_\ell}\kappa(\theta,\theta')u(\theta')d\theta' \Rightarrow \{u_k\}_{k=1}^{K}$$

K largest eigenfunctions of R

$$\boldsymbol{v}_k \triangleq \int_{\Theta_\ell}\boldsymbol{a}(\theta)u_k(\theta)d\theta,$$

- **Special case : raised-cosine kernel** $\kappa(\theta,\theta') = a + b\cos(\omega(\theta-\theta'))$,

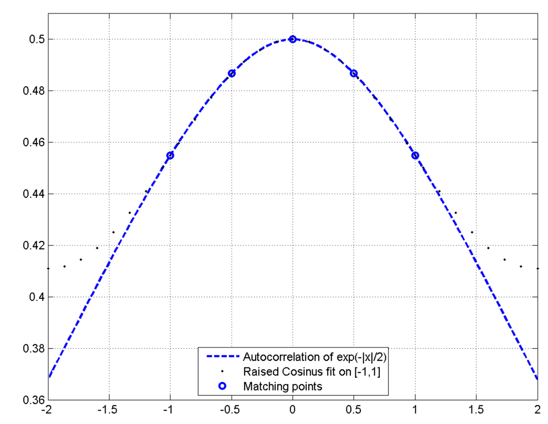
$$\left\{P_{V_{\mathrm{opt}}^3}(\boldsymbol{a}(\theta)) : \theta\in\Theta_\ell\right\} = \left\{\boldsymbol{v}_1 + \cos(\omega\theta)\boldsymbol{v}_2 + \sin(\omega\theta)\boldsymbol{v}_3 : \theta \in \left[-\frac{\Delta}{2},\frac{\Delta}{2}\right]\right\}$$

Polar approximation = ||.||-optimal approximation

Numerical Experiments

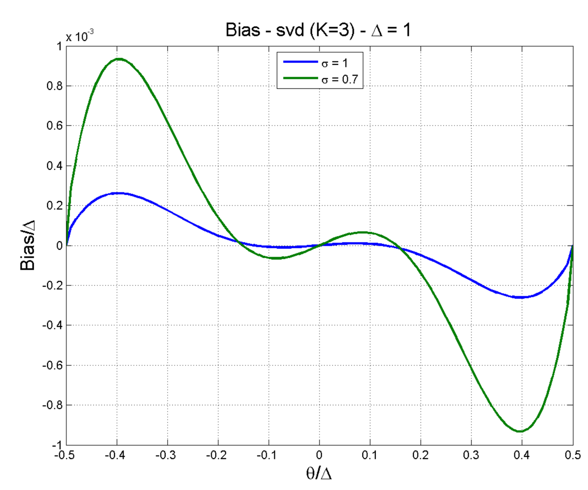
1D Deconvolution : a(θ) = h(. - θ) (h = convolution kernel)
$k(\theta, \theta')$ = $k(\theta - \theta')$ = autocorrelation of h

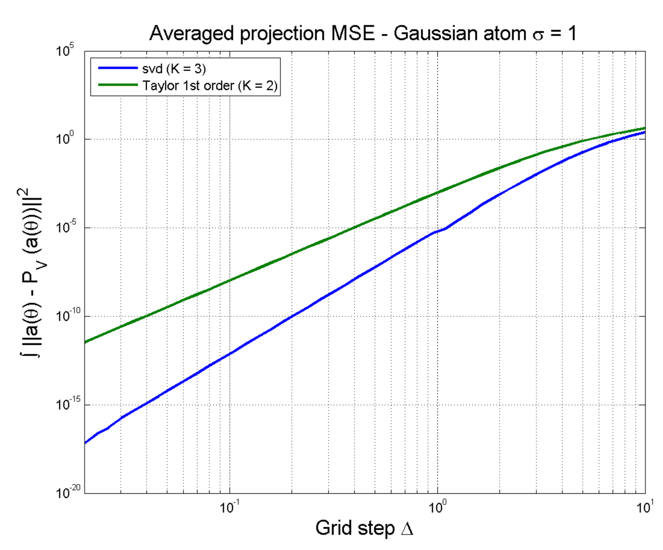
Raised-cosine approximation of $k(\theta - \theta')$
- Most autocorrelations are smooth near zero
- True also for Laplacian kernel despite singularity
- Closed-form expression by maching at 0, $\Delta/2$, Δ

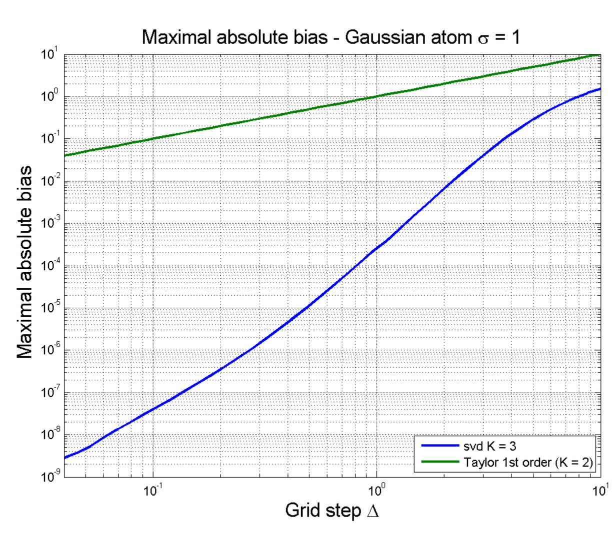


Performance measures
- Bias on location error for ***r*** = **a**(θ^*)
  - Sensitive to location and approximation error
- Averaged Projection MSE

$$\int_{\Theta_\ell}\|\boldsymbol{a}(\theta) - P_V(\boldsymbol{a}(\theta))\|^2 d\theta.$$







SVD / Polar approximation accurate for
- Atom MSE approximation → grid methods (Continuous BP)
- Atom Selection → off the grid methods (Continuous OMP, BLASSO)

Perspectives

- **Towards Polar and SVD approximations in higher dimensions**
- **Coarse to fine strategies for** $\max_{\boldsymbol{a}\in\hat{\mathcal{A}}}\langle \boldsymbol{a},\boldsymbol{r}\rangle = \max_{\ell}\max_{\boldsymbol{a}\in\hat{\mathcal{A}}_\ell}\langle \boldsymbol{a},\boldsymbol{r}\rangle$

References

C. Ekanadham *et al.*, *Recovery of sparse translation-invariant signals with continuous basis pursuit*, TSP, 2011

K. Knudson *et al.*, *Inferring sparse representations of continuous signals with orthogonal matching pursuit*, NIPS, 2014

Q. Denoyelle *et al.*, *The Sliding Frank-Wolfe Algorithm and its Application to Super-Resolution Microscopy*, 2018. (hal-01921604)

C. Herzet *et al.*, *Gather and Conquer: Region-based Strategies to Accelerate Safe Screening Tests*, TIP, 2019