# Waveform Modeling by Adaptive Weighted Hermite Functions

Péter Kovács[1,2] Carl Böck[1,3] Tamás Dózsa[2]
Jens Meier[3] Mario Huemer[1]

[1]Institute of Signal Processing, Johannes Kepler University Linz, Austria
[2]Department of Numerical Analysis, Eötvös Loránd University, Hungary
[3]Department of Anesthesiology and Critical Care Medicine, Kepler University Hospital, Austria

44th IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)

Brighton, UK
May 12-17, 2019

# Outline

1. Introduction
2. Nonlinear model using Hermite functions
3. Case study: ECG compression

# Motivation

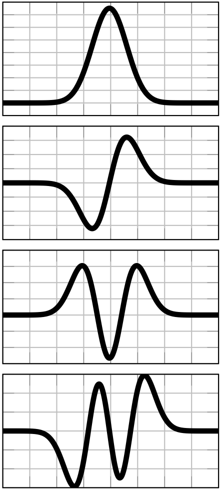
## ECG signal modeling

Due to shape similarities, Hermite functions became very popular in

- modeling QRS shape features,
- ECG data compression,
- clustering QRS complexes,
- detecting abnormalities such as myocardial infarction,
- ECG segmentation and delineation.

## Other applications

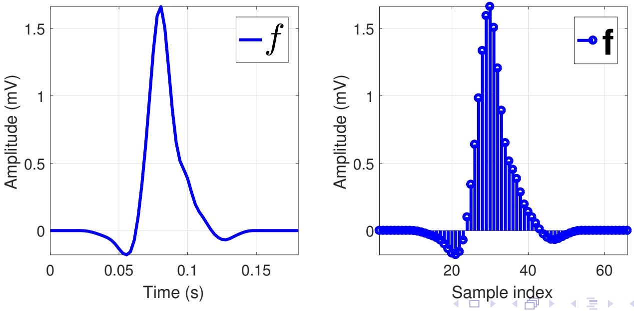
- Ballistocardiogram and myoelectric signal processing,
- image processing,
- computer tomography,
- radar signal processing, and
- physical optics.



# Signal representation

## Notations

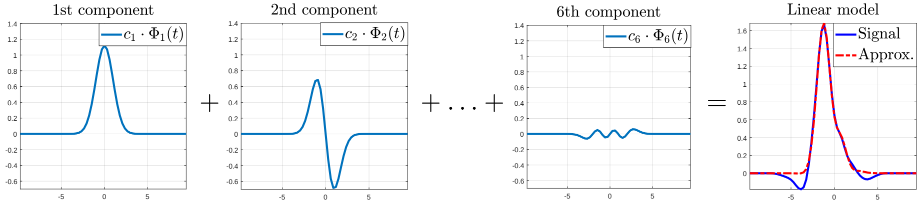
- Continuous signal: $f : \mathbb{R} \to \mathbb{R}$.
- Sampling period: $T$.
- Discrete time instances: $t_i := i \cdot T$.
- Discrete signal having $n$ samples: $[\mathbf{f}]_i := f(t_i)$ for $i = 1, \ldots, n$.



# Linear model

$$[\mathbf{f}]_i = f(t_i) \approx \sum_{k=1}^{m} c_k \Phi_k(t_i) = (\mathbf{\Phi c})_i \quad (i = 1, \ldots, n)$$

- $\{\Phi_k \,|\, 1 \leq k \leq m\}$ can be the set of trigonometric or Walsh functions, wavelets, orthogonal polynomials, splines, etc.
- The least squares (LS) estimate of the coefficients is $\mathbf{c} = \mathbf{\Phi}^{+}\mathbf{f}$, where $\mathbf{\Phi}^{+}$ denotes the Moore–Penrose inverse of $\mathbf{\Phi}$.

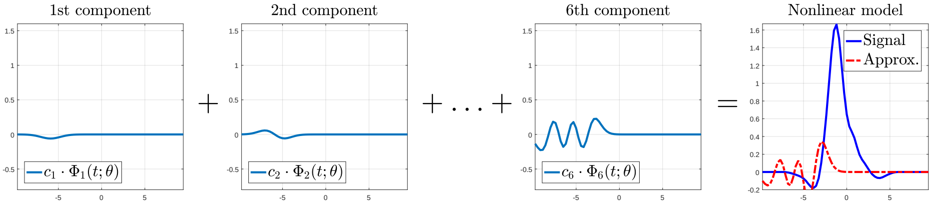



Linear LS approximation of a QRS complex using Hermite functions (no optimization).

# Nonlinear model

$$[\mathrm{f}]_i = f(t_i) \approx \sum_{k=1}^{m} c_k \Phi_k(t_i;\boldsymbol{\theta}) = (\boldsymbol{\Phi}(\boldsymbol{\theta})\boldsymbol{c})_i \quad (i = 1, \ldots, n)$$

- The functions $\{\Phi_k(\cdot;\boldsymbol{\theta}) \,|\, 1 \leq k \leq m\}$ are parametrized by $\boldsymbol{\theta}$.
- $\mathbf{c}$ and $\boldsymbol{\theta}$ are determined via nonlinear optimization.

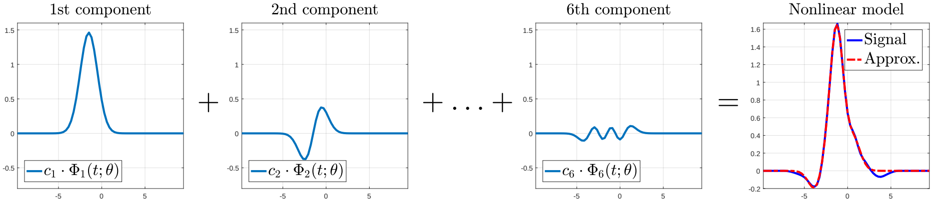



Nonlinear LS approximation of a QRS complex using Hermite functions parametrized by the dilation and the translation.

# Nonlinear model

$$[\mathrm{f}]_i = f(t_i) \approx \sum_{k=1}^{m} c_k \Phi_k(t_i;\boldsymbol{\theta}) = (\boldsymbol{\Phi}(\boldsymbol{\theta})\boldsymbol{c})_i \quad (i = 1, \ldots, n)$$

- The functions $\{\Phi_k(\cdot;\boldsymbol{\theta}) \,|\, 1 \leq k \leq m\}$ are parametrized by $\boldsymbol{\theta}$.
- $\mathbf{c}$ and $\boldsymbol{\theta}$ are determined via nonlinear optimization.




Nonlinear LS approximation of a QRS complex using Hermite functions parametrized by the dilation and the translation (final optimization step).

# Signal model

## Goals

- Generalization of former wave shape models via weight modification.
- Adapt the new weighted Hermite system to various types of signals.
- Heuristics for speeding up the nonlinear optimization.
- Case study: electrocardiogram (ECG) signal compression.

## Related works

- ECG data compression[1]
- ECG segmentation[2]

[1] T. Dózsa and P. Kovács, *ECG signal compression using adaptive Hermite functions*, Advances in Intelligent Systems and Computing, vol. 399, pp. 245–254, 2015.

[2] P. Kovács, C. Böck, J. Meier, M. Huemer, *ECG segmentation using adaptive Hermite functions*, Proceedings of the 51st Asilomar Conference on Signals, Systems, and Computers, 2017, pp. 1476–1480.

# Hermite polynomials

## Terminology

- Three-term recurrence relation:

$$h_{k+1}(t) = (t - \alpha_k)\, h_k(t) - \beta_k h_{k-1}(t), \quad (k \in \mathbb{N}).$$
$$h_{-1}(t) = 0, \qquad h_0(t) = 1\,.$$

- Recurrence coefficients for monic Hermite polynomials:

$$\alpha_k = 0, \quad \beta_0 = \sqrt{\pi}, \quad \beta_k = k/2, \quad (k \in \mathbb{N}^+)\,.$$

- Orthogonality:

$$\|h_k\|_2^2 \cdot \delta_{kj} = \langle h_k, h_j \rangle_w := \int_{-\infty}^{\infty} h_k(t) h_j(t) w(t)\, \mathrm{d}t,$$

where $w(t) = e^{-t^2}$ is the Hermite weight function.

# Hermite functions

## Definition

- System of Hermite polynomials: $\{h_k \mid k \in \mathbb{N}\}$.
- System of Hermite functions: $\{\Phi_k \mid k \in \mathbb{N}\}$, where

$$\Phi_k(t) = h_k(t)/||h_k||_2 \cdot \sqrt{w(t)} \qquad (k \in \mathbb{N}).$$

## Affine argument transform

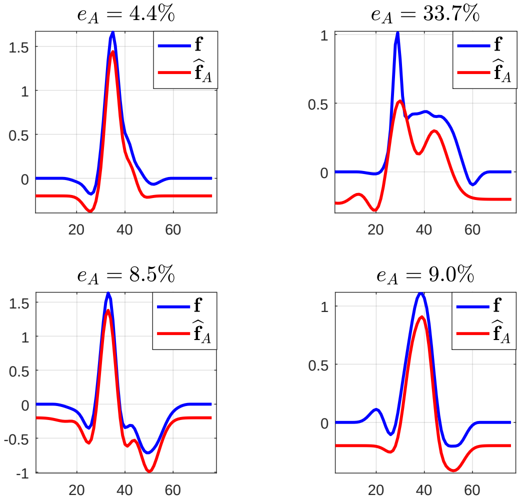
$$\Phi_k(t;\tau;\lambda) := \sqrt{\lambda}\Phi_k(\lambda(t-\tau)) \qquad (t,\tau \in \mathbb{R}, \lambda > 0).$$

## Optimization problem

Find $\lambda$, $\tau$ that minimize the so-called variable projection functional

$$r_2(\lambda;\tau) = \|\mathbf{f} - \mathbf{\Phi}(\lambda;\tau)\mathbf{\Phi}^+(\lambda;\tau)\mathbf{f}\|_2^2 = \|\mathbf{f} - \mathbf{P}_{\Phi(\lambda;\tau)}\mathbf{f}\|_2^2.$$

# Hermite functions




Examples using 6 basis functions for which the affine Hermite model (red line) does not work perfectly.

# Weighted Hermite functions

## Modifying the weight function

- Let us define the class of weight functions:

$$\mathcal{V} = \left\{ v(\cdot;\boldsymbol{\eta}) \in C(\mathbb{R}) : v \geq 0,\ \exists \gamma > 0,\ \sup_{t\in\mathbb{R}} |v(t;\boldsymbol{\eta})| e^{\gamma t^2} < \infty \right\}.$$

- $\{q_k \,|\, k \in \mathbb{N}\}$ is the set of orthogonal polynomials defined by $v \in \mathcal{V}$.
- Then the weighted Hermite functions are defined as follows:

$$\Psi_k(t;\boldsymbol{\eta}) = q_k(t)/||q_k||_2 \cdot \sqrt{v(t;\boldsymbol{\eta})} \qquad (k \in \mathbb{N},\ v \in \mathcal{V}, t \in \mathbb{R}).$$

## Restrictions of $\mathcal{V}$

We consider nonnegative weight functions of the form

$$v(t;\boldsymbol{\eta}) = u(t;\boldsymbol{\eta}) \cdot w(t) = p_1(t;\boldsymbol{\eta})/p_2(t;\boldsymbol{\eta}) \cdot e^{-t^2},$$

where $p_1, p_2$ are polynomials in $t$ of degree $\ell, m$ such that $p_1/p_2 \geq 0$.

# Weighted Hermite functions

## Modification algorithms

- Due to the partial fraction decomposition of $u(t, \boldsymbol{\eta})$, it suffices to consider the factors $t - \eta_1$ and $(t-\eta_1)^2 + \eta_2^2$ and analogous divisors.
- Since the Hermite functions are defined over $\mathbb{R}$, only the following elementary modifications and their finite sums are allowed:

$$v_1(t;\eta_1) := u_1(t;\eta_1) \cdot w(t), \quad u_1(t;\eta_1) := (t-\eta_1)^2,$$
$$v_2(t;\boldsymbol{\eta}) := u_2(t;\boldsymbol{\eta}) \cdot w(t), \quad u_2(t;\boldsymbol{\eta}) := 1/((t-\eta_1)^2 + \eta_2^2),$$
$$(t \in \mathbb{R},\ (\eta_1, \eta_2) \in \mathbb{R}^2,\ \eta_2 \neq 0,\ v_1, v_2 \in \mathcal{V}).$$

## Full problem (optimal weighting + affine trf.)

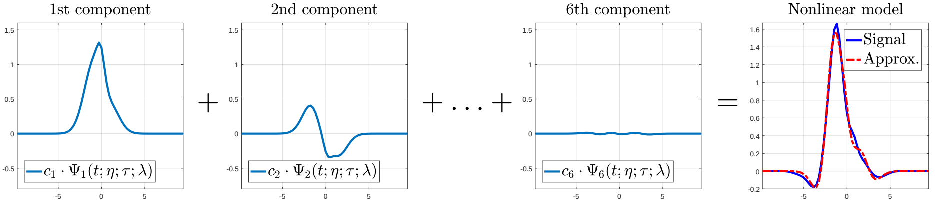
- $\boldsymbol{\Psi}(\boldsymbol{\eta};\tau;\lambda)_{ik} := \Psi_k(t_i;\boldsymbol{\eta};\tau;\lambda) = \Psi_k(\lambda(t_i-\tau);\boldsymbol{\eta})$
- The extended variable projection problem can be written as:

$$\min_{\boldsymbol{\eta},\tau,\lambda} r_2(\boldsymbol{\eta};\tau;\lambda) = \min_{\boldsymbol{\eta},\tau,\lambda} \|\mathbf{f} - \boldsymbol{\Psi}(\boldsymbol{\eta};\tau;\lambda)\boldsymbol{\Psi}^+(\boldsymbol{\eta};\tau;\lambda)\mathbf{f}\|_2^2.$$

# Weighted Hermite functions

## Difficulties

- The system of weighted Hermite functions $\{\Psi_k(\cdot;\boldsymbol{\eta}) \mid k \in \mathbb{N}\}$ depends on the nonlinear parameters $\boldsymbol{\eta}$.
- Therefore, recomputing the corresponding recurrence coefficients $\hat{\alpha}_k(\boldsymbol{\eta})$ and $\hat{\beta}_k(\boldsymbol{\eta})$ for each value of $\boldsymbol{\eta}$ is a difficult task.

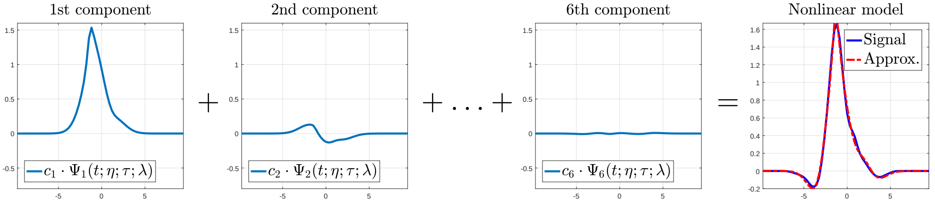



Full problem: nonlinear LS approximation of a QRS complex using translated and dilated weighted Hermite functions.

# Weighted Hermite functions

## Difficulties

- The system of weighted Hermite functions $\{\Psi_k(\cdot;\boldsymbol{\eta}) \mid k \in \mathbb{N}\}$ depends on the nonlinear parameters $\boldsymbol{\eta}$.
- Therefore, recomputing the corresponding recurrence coefficients $\hat{\alpha}_k(\boldsymbol{\eta})$ and $\hat{\beta}_k(\boldsymbol{\eta})$ for each value of $\boldsymbol{\eta}$ is a difficult task.




Full problem: nonlinear LS approximation of a QRS complex using translated and dilated weighted Hermite functions (final optimization step).

# Weighted Hermite functions

## Idea

Reduce the full optimization problem to two simple sub-tasks.

## Reduced problem

1. Instead of the full system, optimize the weight function only:

$$\min_{\boldsymbol{\eta},\tau,\lambda} r_2(\boldsymbol{\eta};\tau;\lambda) = \min_{\boldsymbol{\eta},\tau,\lambda} \|\mathbf{f} - \sqrt{\mathbf{v}(\boldsymbol{\eta};\tau;\lambda)}\|_2^2,$$

   where $\mathbf{v}(\boldsymbol{\eta};\tau;\lambda)_i := v(t_i;\boldsymbol{\eta};\tau;\lambda) = v(\lambda(t_i-\tau);\boldsymbol{\eta})$.

2. Then, fix $\boldsymbol{\eta}$, and find the best affine parameters:

$$\min_{\tau,\lambda} r_2(\boldsymbol{\eta};\tau;\lambda) = \min_{\tau,\lambda} \|\mathbf{f} - \boldsymbol{\Psi}(\boldsymbol{\eta};\tau;\lambda)\boldsymbol{\Psi}^+(\boldsymbol{\eta};\tau;\lambda)\mathbf{f}\|_2^2.$$

# Weighted Hermite functions

## Numerical optimization[1]

- Projection operator: $\mathbf{P}^{\perp}_{\Psi(\boldsymbol{\eta})} := \mathbf{I} - \mathbf{P}_{\Psi(\boldsymbol{\eta})} = \mathbf{I} - \boldsymbol{\Psi}(\boldsymbol{\eta})\boldsymbol{\Psi}^{+}(\boldsymbol{\eta})$.
- For the sake of simplicity, we omit the vector of free parameters $\boldsymbol{\eta}$ from the notations. Then, the $j$th coordinate of the gradient is

$$\frac{1}{2}\nabla r_2^{(j)} = \left(-\left(\mathbf{P}^{\perp}_{\Psi}\mathbf{D}_j\boldsymbol{\Psi}^{+} + \left(\mathbf{P}^{\perp}_{\Psi}\mathbf{D}_j\boldsymbol{\Psi}^{+}\right)^T\right)\mathbf{f}\right)^T \mathbf{P}^{\perp}_{\Psi}\mathbf{f}\,,$$

  where $\mathbf{D}_j := \partial\boldsymbol{\Psi}(\boldsymbol{\eta})/\partial\eta_j$.
- It can be calculated only for the affine and the reduced problem.
- In case of the full problem, quasi-Newton methods can be used.

[1] G. H. Golub and V. Pereyra, *The differentiation of pseudo-inverses and nonlinear least squares problems whose variables separate*, SIAM Journal on Numerical Analysis (SINUM), vol. 10, pp. 413–432, 1973.

# Constraining the optimization

## Parameters of $u_1(t;\eta) = (t-\eta)^2$

If $\eta$ is far from zero, $u_1$ is smoothed out by the tails of $w(t) = e^{-t^2}$.

Since $w$ is a Gaussian, the three-sigma rule applies with $\sigma = 1/\sqrt{2}$.

Therefore, we restrict the values of $\eta$ to the interval $[-3\sigma; 3\sigma]$.

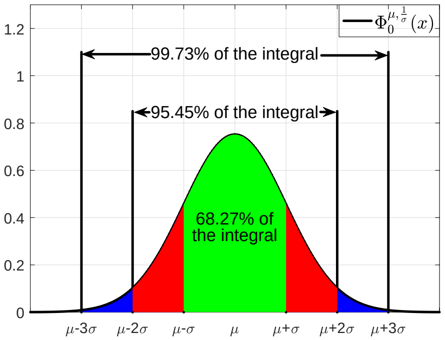
## Parameters of $u_2(t;\boldsymbol{\eta}) = 1/((t-\eta_1)^2 + \eta_2^2)$

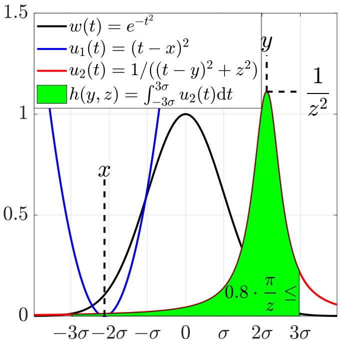
Choose $\boldsymbol{\eta}$ such that the following inequality is satisfied:

$$0.8 \cdot \int_{-\infty}^{\infty} u_2(t;\boldsymbol{\eta})\,\mathrm{d}t \le \int_{-3\sigma}^{3\sigma} u_2(t;\boldsymbol{\eta})\,\mathrm{d}t.$$

It means that the main lobe of $u_2$ cannot be too wide, i.e., 80% of its overall integral should lie in the interval $[-3\sigma; 3\sigma]$.

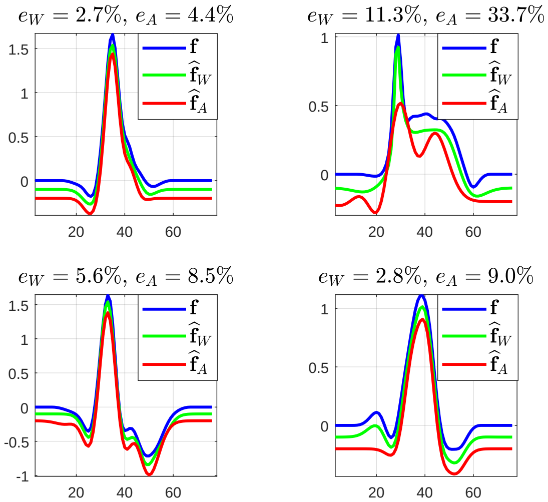
# Constraining the optimization




The three-sigma rule and the constraints.

# Weighted Hermite functions - Illustrations

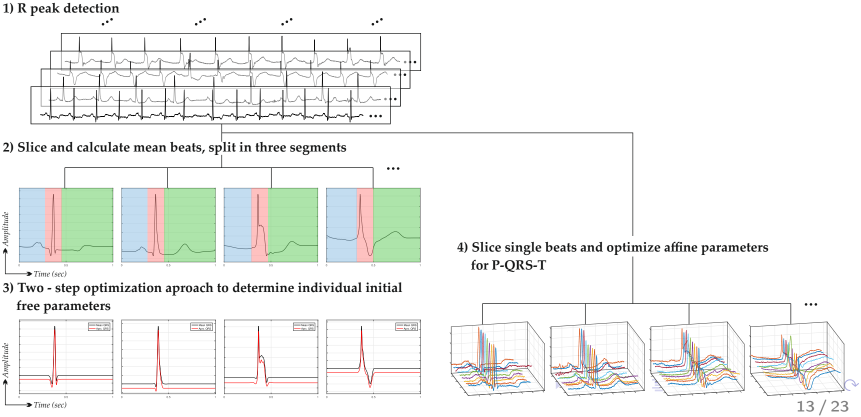



Examples using 6 basis functions for which the affine Hermite model (red line) does not work perfectly, the weighted Hermite model (green line) performs better.

# Experiments

## Evaluation method and data

- ECG data compression.
- Good benchmark regarding the distortion of ECG signals.
- 12 hours of ECG raw data from MIT/BIH arrhythmia database.



# Experiments cont.

## Evaluation method and data

- ECG data compression
- Good benchmark regarding the distortion of ECG signals
- 12 hours of ECG raw data from MIT/BIH arrhythmia database

## Comparison to other work

- Compression ratio
- Normalized PRDN for M ECG beats / QRS complexes

$$\overline{\mathrm{PRDN}} = 100 \cdot \frac{1}{M} \sum_{m=1}^{M} \frac{||\mathbf{f_m} - \hat{\mathbf{f}}_\mathbf{m}||_2}{||\mathbf{f_m} - \bar{\mathbf{f}}_\mathbf{m}||_2},$$

- Dilation only vs. affine transformation vs. our work

# Results

Experimental results of 12 hours long real ECG data.

| | Dilation only[1] | | | Affine trf.[2] | | | Proposed work | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Rec. | $\overline{\text{PRDN}}$ | $\frac{\text{QRS}}{\text{PRDN}}$ | CR | $\overline{\text{PRDN}}$ | $\frac{\text{QRS}}{\text{PRDN}}$ | CR | $\overline{\text{PRDN}}$ | $\frac{\text{QRS}}{\text{PRDN}}$ | CR | System |
| **mean** | **18.18** | **20.28** | **19.75** | **15.40** | **11.40** | **18.83** | **14.88** | **9.86** | **18.82** | - |
| Selected recordings (for illustration) | | | | | | | | | | |
| 100 | 17.09 | 16.57 | 19.47 | 13.09 | 12.09 | 18.52 | 9.78 | 7.74 | 18.52 | qf |
| 102 | 33.57 | 36.34 | 20.22 | 33.69 | 24.91 | 19.23 | 31.27 | 16.05 | 19.23 | qd |
| 104 | 31.94 | 39.49 | 19.96 | 34.10 | 37.57 | 18.98 | 29.78 | 20.38 | 18.98 | qd |
| 232 | 32.40 | 20.28 | 24.39 | 26.00 | 14.55 | 23.22 | 24.18 | 9.59 | 23.21 | qd+qf |

[1] R. Jané, S. Olmos, P. Laguna, and P. Caminal, *Adaptive Hermite models for ECG data compression: performance and evaluation with automatic wave detection*, in Proc. of Computers in Cardiology Conference, 1993, pp. 389–392.

[2] T. Dózsa and P. Kovács, *ECG signal compression using adaptive Hermite functions*, Advances in Intelligent Systems and Computing, vol. 399, pp. 245–254, 2015.

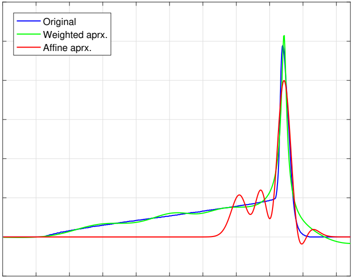
# Conclusions and applications

## Conclusions

- Generalization of former wave shape models allowing to model more complex wave forms
- Preoptimized parameters $\rightarrow$ subject-specific.
- $\mathbf{c}$, $\tau$, $\lambda$ $\rightarrow$ morphological changes over time.

## Applications

- Signal classification and detection, information extraction.
- Potentially suitable for modeling action potentials, blood pressure, or other biomedical signals.



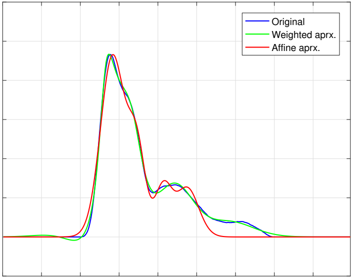
# Conclusions and applications

## Conclusions

- Generalization of former wave shape models allowing to model more complex wave forms
- Preoptimized parameters $\rightarrow$ subject-specific.
- $\mathbf{c}$, $\tau$, $\lambda$ $\rightarrow$ morphological changes over time.

## Applications

- Signal classification and detection, information extraction.
- Potentially suitable for modeling action potentials, blood pressure, or other biomedical signals.



# Waveform Modeling by Adaptive Weighted Hermite Functions

Péter Kovács[1,2] Carl Böck[1,3] Tamás Dózsa[2]
Jens Meier[3] Mario Huemer[1]

[1]Institute of Signal Processing, Johannes Kepler University Linz, Austria
[2]Department of Numerical Analysis, Eötvös Loránd University, Hungary
[3]Department of Anesthesiology and Critical Care Medicine, Kepler University Hospital, Austria

44th IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)

Brighton, UK
May 12-17, 2019

# Conclusions cont.

## Advantages compared to PCA, DWT, etc.

- Method works for ECG recordings of any size (PCA not suited for short recordings).
- Domain of translation / dilation is continuous $\rightarrow$ resampling at any grid is possible.
- Wide field of applications (biomedical / image / radar signal processing, computer tomography, physical optics, ...).
- Automatic separation of morphological changes induced by translation ($\tau$), dilation ($\lambda$), or other sources (e.g. change of amplitude $\rightarrow$ $\mathbf{c}$). Methods like PCA lack this ability.

# Separation of morphological changes

## Translation

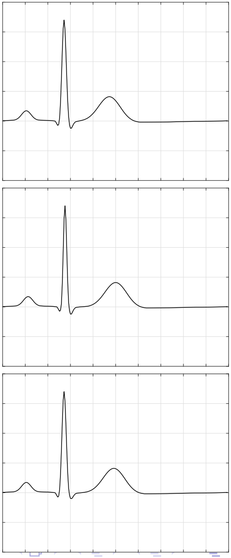
Morphological changes induced by the translation of a wave (e.g. due to changing heart rate) are captured by $\tau$.

## Dilation

Morphological changes induced by the dilation of a wave are captured by $\lambda$.

## Additional Morphological changes

Morphological changes which are not induced by translation or dilation but other (possibly diagnostic) sources, should be captured by $\mathbf{c}$.



# References for orthogonal polynomials


- G. Szegő, *Orthogonal Polynomials*, AMS Colloquium Publications, New York, USA, 3rd edition, 1967.
- W. Gautschi, *Orthogonal Polynomials, Computation and Approximation*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, UK, 2004.
- B. Fischer and G. H. Golub, *How to generate unknown orthogonal polynomials out of known orthogonal polynomials*, Computation and Approximation, Journal of Computational and Applied Mathematics, vol. 43, pp. 99–115, 1992.
- B. Fischer and G. H. Golub, *On generating polynomials which are orthogonal over several intervals*, Mathematics of Computation, vol. 56, pp. 711–730, 1991.

# References for ECG modeling


- L. Sörnmo, P. L. Börjesson, M. E. Nygårds, and O. Pahlm, *A method for evaluation of QRS shape features using a mathematical model for the ECG*, IEEE Transactions on Biomedical Engineering, vol. 28, pp. 713–717, 1981.
- R. Jané, S. Olmos, P. Laguna, and P. Caminal, *Adaptive Hermite models for ECG data compression: performance and evaluation with automatic wave detection*, Proceedings of Computers in Cardiology Conference, 1993, pp. 389-392.
- T. Dózsa and P. Kovács, *ECG signal compression using adaptive Hermite functions*, Advances in Intelligent Systems and Computing, vol. 399, pp. 245–254, 2015.
- P. Kovács, C. Böck, J. Meier, and M. Huemer, *ECG segmentation using adaptive Hermite functions*, in Proceedings of the 51st Asilomar Conference on Signals, Systems, and Computers, 2017, pp. 1476–1480.

# References for ECG modeling


- H. Haraldsson, L. Edenbrandt, and M. Ohlsson, *Detecting acute myocardial infarction in the 12-lead ECG using Hermite expansions and neural networks*, Artificial Intelligence in Medicine, vol. 32, pp. 127–136, 2004.
- A. Sandryhaila, S. Saba, M. Püschel, and J. Kovacevic, *Efficient compression of QRS complexes using Hermite expansion*, IEEE Transactions on Signal Processing, vol. 60, no. 2, pp. 947–955, 2012.
- M. Lagerholm, C. Peterson, G. Braccini, L. Edenbrandth, and L. Sörnmo, "Clustering ECG complexes using Hermite functions and self-organizing maps," *IEEE Transactions on Biomedical Engineering*, vol. 47, no. 7, pp. 838–717, 2000.

# References for other applications


- A. Mahadevan, S. Acharya, D. B. Sheffer, and D. H. Mugler, *Ballistocardiogram Artifact Removal in EEG-fMRI Signals Using Discrete Hermite Transforms*, IEEE Journal of Selected Topics in Sig. Proc., vol. 2, pp. 839–853, 2008.
- G. Leibon, D. N. Rockmore, W. Park, R. Taintor, G. S. Chirikjian, *A fast Hermite transform*, Theor. Comp. Sci., vol. 409, pp. 211–228, 2008.
- E. Moya-Albora, B. Escalante-Ramíreza, E. Vallejob, *Optical flow estimation in cardiac CT images using the steered Hermite transform*, Sig. Proc.:Image Commun., vol. 28, pp. 267–291, 2013.
- S. Stanković, I. Orović, A.Krylov, *The two-dimensional Hermite S-method for high resolution inverse synthetic aperture radar imaging applications*, IET Sig. Proc., vol. 4, pp. 352-362, 2010.
- P. Lazaridis, G. Debarge, P. Gallion, *Discrete orthogonal Gauss–Hermite transform for optical pulse propagation analysis*, J. Opt. Soc. Am. B, vol. 20, pp. 1508–1513, 2003.