# Tropical Modeling of Transducer Algorithms on Graphs


*Emmanouil Theodosis, Petros Maragos*

School of Electrical and Computer Engineering
National Technical University of Athens, Greece


group
paper


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**CVSP:** http://cvsp.cs.ntua.gr **IRAL:** https://robotics.ntua.gr

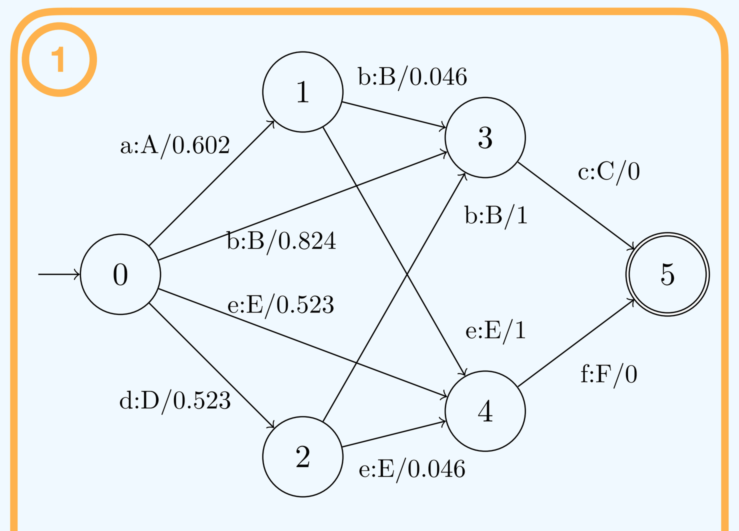

## 1. Introduction

***Why WFST?***

- extensive application in SLP
- tropical operations, but conventional calculations

***Why tropical?***

- emerging field with novel results
- piecewise linear solution space
- connection to tropical geometry
- nonlinear vector spaces
- unified common framework




Weighted Finite State Transducer

## 2. Tropical Algebra and Geometry

- like linear algebra, but (+, x) ⟶ (∧, +) min
- matrix/vector multiplication:

$$(\mathbf{A} \boxplus \mathbf{B})_{ij} = \bigwedge_{k=1}^{n} A_{ik} + B_{kj}$$

- neutral elements are ∞ for the minimum and 0 for the addition
- example:

$$\begin{bmatrix} \infty & 4 \\ -6 & 11 \end{bmatrix} \boxplus \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} \min(\infty+7, 4+0) \\ \min(-6+7, 11+0) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

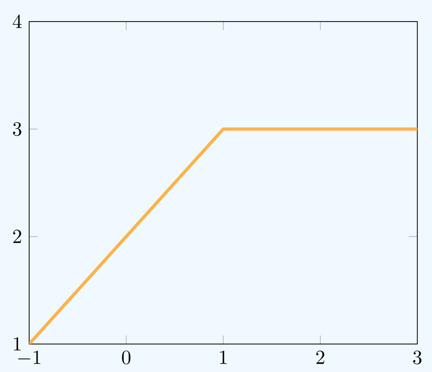
- transitive closures:

$$\Gamma(\mathbf{A}) = \mathbf{A} \wedge \mathbf{A}^2 \wedge \ldots \wedge \mathbf{A}^n \wedge \ldots$$
$$\Delta(\mathbf{A}) = \mathbf{I} \wedge \mathbf{A} \wedge \mathbf{A}^2 \wedge \ldots \wedge \mathbf{A}^n \wedge \ldots$$

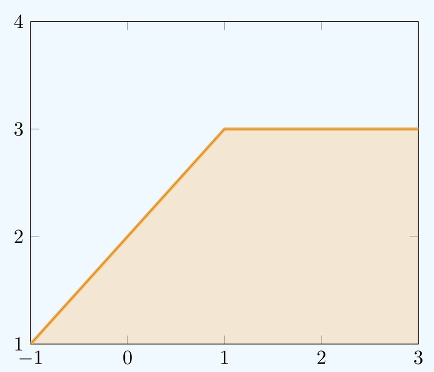
solutions to eigenvalue problems

- tropical line:

$$y = \alpha + x \wedge \beta = \min(\alpha + x, \beta)$$



tropical line
$y = 2 + x \wedge 3$



tropical halfspace
$y \leq 2 + x \wedge 3$

polytope: bounded intersection of halfspaces

## 3. Modeling

### 3.1 *Weight pushing*

- single step update:

$$\mathbf{v}_{i+1} = \mathbf{v}_i \wedge \mathbf{A} \boxplus \mathbf{v}_i$$

- final potential vector:

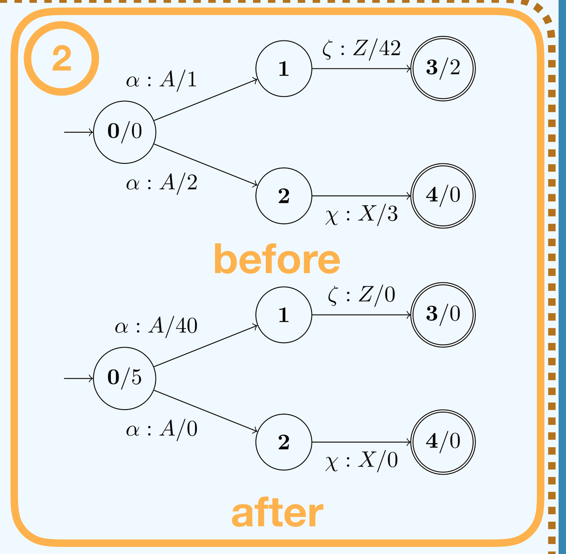
$$\mathbf{v}_\infty = \rho \wedge \mathbf{A} \boxplus \rho \wedge \mathbf{A}^2 \boxplus \rho \wedge \ldots = \Delta(\mathbf{A}) \boxplus \rho$$

- define:

$$\boldsymbol{\Lambda} = \mathbf{diag}(\lambda)$$
$$\mathbf{V}^+ = \mathbf{diag}(\mathbf{v}_\infty)$$
$$\mathbf{V}^- = \mathbf{diag}(-\mathbf{v}_\infty)$$
$$\mathbf{P} = \mathbf{diag}(\rho)$$

- new system model:

$$\lambda' = \boldsymbol{\Lambda} \boxplus \mathbf{v}_\infty, \quad \rho' = \mathbf{P} \boxplus (-\mathbf{v}_\infty), \quad \mathbf{A}' = \mathbf{V}^- \boxplus \mathbf{A} \boxplus \mathbf{V}^+$$




**A**: transition matrix
**λ**: input vector
**ρ**: emmission vector

### 3.2 *Epsilon removal*

- decompose:

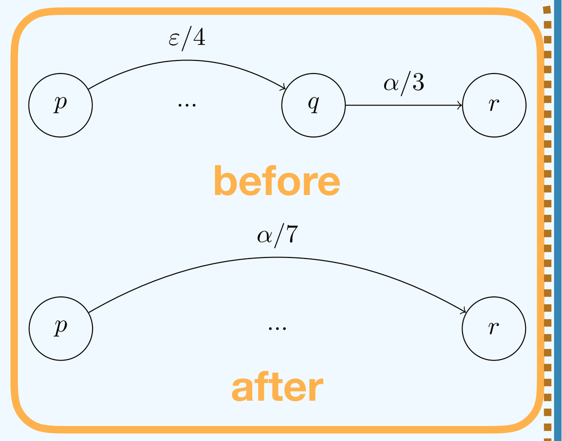
$$\mathbf{A} = \mathbf{A}_\epsilon \wedge \mathbf{E}$$

- epsilon closure of WFST:

$$\Gamma(\mathbf{E}) = \mathbf{E} \wedge \mathbf{E}^2 \wedge \ldots \wedge \mathbf{E}^n \wedge \ldots$$

- new system model:

$$\mathbf{A}' = \mathbf{A}_\epsilon \wedge (\Gamma(\mathbf{E}) \boxplus \mathbf{A}_\epsilon) = \Delta(\mathbf{E}) \boxplus \mathbf{A}_\epsilon$$
$$\rho' = \rho \wedge (\Gamma(\mathbf{E}) \boxplus \rho) = \Delta(\mathbf{E}) \boxplus \rho$$




**$\Sigma_I$**: input symbols
**$\Sigma_O$**: output symbols

### 3.3 *Viterbi pruning*

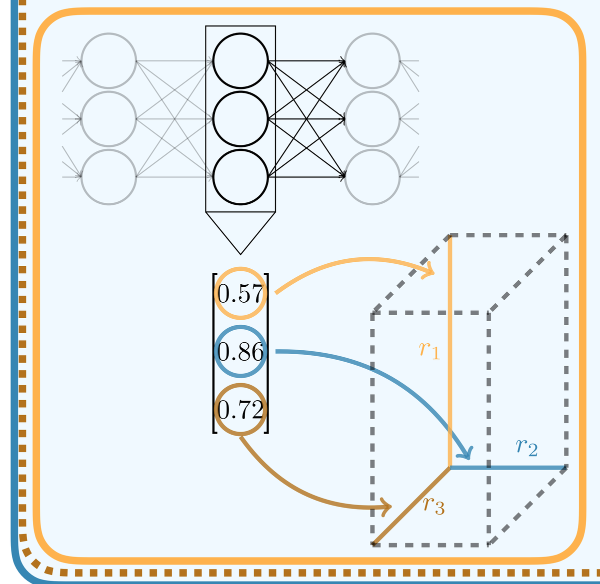
- tropical Viterbi:

$$\mathbf{x}(t) = \mathbf{P}(\sigma_t) \boxplus \mathbf{A}^T \boxplus \mathbf{x}(t-1)$$

- variable vector **z**:

$\mathbf{z} \geq \mathbf{b}$ → Viterbi update
$\mathbf{z} \leq \eta$ → Viterbi update + threshold

$$\mathbf{b} = \mathbf{P}(\sigma_t) \boxplus \mathbf{A}^T \boxplus \mathbf{x}(t-1)$$
$$\eta = \theta + \frac{1}{2}(\mathbf{b}^T \boxplus \mathbf{b}) + \mathbf{0}$$
$$r_i = \eta - z_i$$



- volume metric:

$$\nu = -\frac{1}{|\,\mathbf{supp}(\mathbf{z})\,|} \sum_{i \in \mathbf{supp}(\mathbf{z})} \frac{\log r_i}{\log(\max \mathbf{r})}$$

- entropy metric:

$$\varepsilon = -\frac{1}{|\,\mathbf{supp}(\mathbf{z})\,|} \sum_{i \in \mathbf{supp}(\mathbf{z})} -z_i(t) \cdot e^{-z_i(t)}$$

## 4. Examples

***Weight pushing*** 2

- original model:

$$\mathbf{A} = \begin{bmatrix} \infty & 1 & 2 & \infty & \infty \\ \infty & \infty & \infty & 42 & \infty \\ \infty & \infty & \infty & \infty & 3 \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 \\ \infty \\ \infty \\ \infty \\ \infty \end{bmatrix}, \quad \rho = \begin{bmatrix} \infty \\ \infty \\ \infty \\ 2 \\ 0 \end{bmatrix}$$

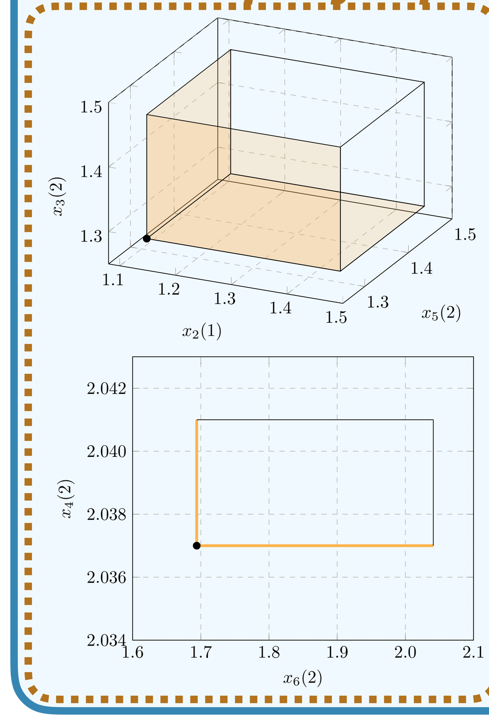
- final potential vector:

$$\Delta(\mathbf{A}) = \begin{bmatrix} 0 & 1 & 2 & 43 & 5 \\ \infty & 0 & \infty & 42 & \infty \\ \infty & \infty & 0 & \infty & 3 \\ \infty & \infty & \infty & 0 & 0 \\ \infty & \infty & \infty & \infty & 0 \end{bmatrix}, \quad \mathbf{v}_\infty = \begin{bmatrix} 5 \\ 44 \\ 3 \\ 2 \\ 0 \end{bmatrix}$$

- new model:

$$\mathbf{A} = \begin{bmatrix} \infty & 40 & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}, \quad \lambda = \begin{bmatrix} 5 \\ \infty \\ \infty \\ \infty \\ \infty \end{bmatrix}, \quad \rho = \begin{bmatrix} \infty \\ \infty \\ \infty \\ 0 \\ 0 \end{bmatrix}$$

***Viterbi polytopes*** 1




- system model:

$$\mathbf{A} = \begin{bmatrix} \infty & 0.602 & 0.523 & 0.824 & 0.523 & \infty \\ \infty & \infty & \infty & 0.046 & 1 & \infty \\ \infty & \infty & \infty & 1 & 0.046 & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

- Viterbi state vectors:

$\theta = 0.35$

$$\mathbf{x}(1) = \begin{bmatrix} \infty \\ 1.125 \\ 1.28 \\ 1.581 \\ 1.28 \\ \infty \end{bmatrix}, \quad \mathbf{x}(2) = \begin{bmatrix} \infty \\ \infty \\ \infty \\ 1.694 \\ 2.083 \\ 2.037 \end{bmatrix}, \quad \mathbf{x}(3) = \begin{bmatrix} \infty \\ \infty \\ \infty \\ \infty \\ \infty \\ 2.217 \end{bmatrix}$$

## 5. Conclusion

***Contributions***

- modeling WFST algorithms in tropical algebra
- analyzing the geometry of Viterbi pruning
- unifying WFST algorithms under a common framework

***Future work***

- interpret the algorithms as tropical eigenvalue problems
- consider alternate bases
- model intrusive algorithms in tropical algebra

## 6. References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