DOMAIN ADAPTATION USING RIEMANNIAN GEOMETRY OF SPD MATRICES

SIPL Signal and Image Processing Lab

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Goal

- **Goal:** Feature extraction and classification of a dataset with multiple domains.
- **Objective:** Obtain a new representation using an unsupervised domain adaptation method based on parallel transport (PT) and moments alignment.
- **Applicability:** Domain adaptation and classification of high-dimensional noisy electrophysiological (EEG) signals collected from different subjects, where each subject is considered as a different domain.

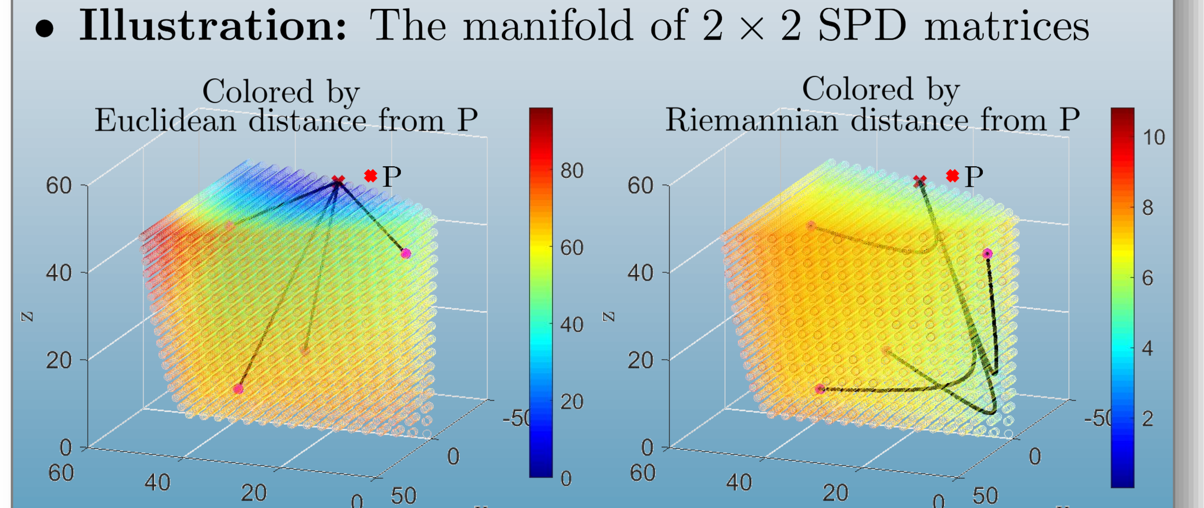
Problem Formulation

- Let $\{X_i^{(1)}, y_i^{(1)}\}_{i=1}^{N_1}$ be the source set and let $\{X_i^{(2)}, y_i^{(2)}\}_{i=1}^{N_2}$ be the target set.

  Each set $k = 1, 2$ contains N_k matrices of data observations $X_i^{(k)} \in \mathbb{R}^{d \times T_i^{(k)}}$, such that d is the observation dimension, $T_i^{(k)}$ is the observation length and $y_i^{(k)}$ is the corresponding label.
- Let $P_i^{(k)} \in \mathbb{R}^{d \times d}$ denote the sample covariance of the data matrix $X_i^{(k)}$, given by $P_i^{(k)} \triangleq \frac{1}{T_i^{(k)}} X_i^{(k)} \left(X_i^{(k)}\right)^T$, where for simplicity we assume zero mean.
- We aim to find a new representation for $\{P_i^{(1)}\}$ and $\{P_i^{(2)}\}$, that will allow to train a classifier on $\{P_i^{(1)}\}$ and to apply it to $\{P_i^{(2)}\}$ (domain adaptation).

SPD Matrices

- A symmetric matrix $P \in \mathbb{R}^{d \times d}$ is positive-definite (SPD) if all its eigenvalues are strictly positive, or equivalently, if $v^T P v > 0$ for every $v \neq 0$.
- The set of all SPD matrices is an open convex cone, constituting a differential Riemannian manifold $\mathcal{M}$.
- **Illustration:** The manifold of 2×2 SPD matrices



Each point (x, y, z) represents a matrix $M = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$.
$M > 0$ iff $x > 0, z > 0$ and $y^2 < xz$.

The 4 magenta points represent matrices with equal Riemannian distance from P. The curves are the Euclidean and Riemannian geodesic respectively.

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The Riemannian Cone Manifold of SPD Matrices

- Let $\mathcal{T}_P\mathcal{M}$ be the tangent space at the point $P \in \mathcal{M}$, equipped with the following inner product:

$$\left\langle S_1, S_2 \right\rangle_{\mathcal{T}_P\mathcal{M}} = \left\langle P^{-\frac{1}{2}} S_1 P^{-\frac{1}{2}}, P^{-\frac{1}{2}} S_2 P^{-\frac{1}{2}} \right\rangle$$

  where $S_1, S_2 \in \mathcal{T}_P\mathcal{M}$ are two symmetric matrices.
- Let $P_1, P_2 \in \mathcal{M}$; the Riemannian geodesic distance is defined by:

$$\delta_R(P_1, P_2) \triangleq \left\| \log\left(P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}}\right) \right\|_F$$

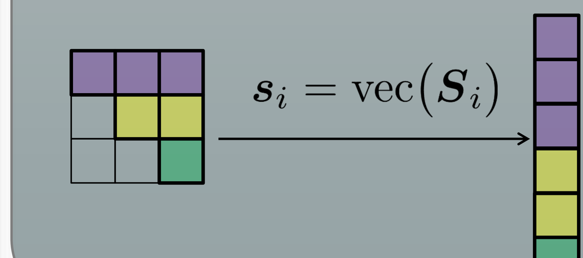
- The Riemannian mean of a set $\{P_i \in \mathcal{M}\}_{i=1}^N$ is defined using the Fréchet mean:

$$\overline{P} \triangleq \arg\min_{P \in \mathcal{M}} \sum_{i=1}^N \delta_R^2(P, P_i)$$

- The logarithmic mapping from $P_i \in \mathcal{M}$ to $S_i \in \mathcal{T}_P\mathcal{M}$ is defined by:

$$S_i = \mathrm{Log}_P(P_i) \triangleq P^{\frac{1}{2}} \log\left(P^{-\frac{1}{2}} P_i P^{-\frac{1}{2}}\right) P^{\frac{1}{2}}$$

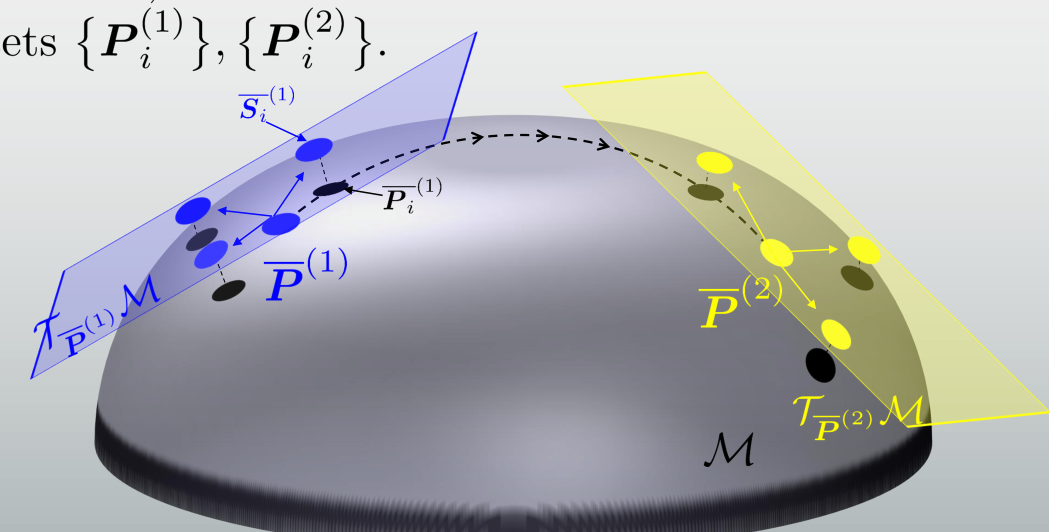
- We denote the vector representation $s_i = \mathrm{vec}(S_i)$



Where $\mathrm{vec}(\cdot)$ is the vectorized upper triangular of a symmetric matrix, with $\sqrt{2}$ weights for its off-diagonal elements.

Parallel Transport (PT)

- Let $\overline{P}^{(1)}, \overline{P}^{(2)}$ be the Riemannian means of the two sets $\{P_i^{(1)}\}, \{P_i^{(2)}\}$.



- The PT from $\overline{P}^{(1)}$ to $\overline{P}^{(2)}$ of any SPD matrix in $\{P_i^{(1)}\}$ is given by:

$$\Gamma_i^{(1)} = \Gamma_{\overline{P}^{(1)} \to \overline{P}^{(2)}}\left(P_i^{(1)}\right) = E P_i^{(1)} E^T$$

  where $E = \left(\overline{P}^{(2)} \left(\overline{P}^{(1)}\right)^{-1}\right)^{\frac{1}{2}}$.

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PT and Moments Alignment

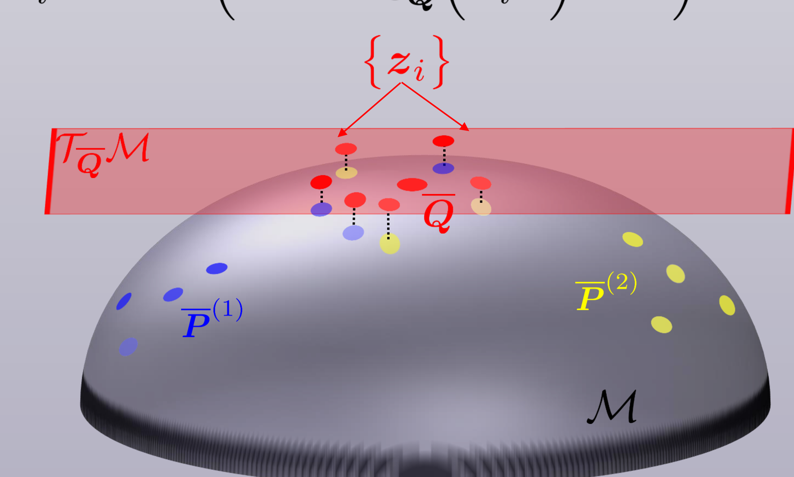
Algorithm 1

Input: K sets $\left\{P_i^{(k)} \in \mathbb{R}^{d \times d}\right\}_{i=1}^{N_k}$, $k = 1, \ldots, K$.

Output: K matrices $\left\{Y^{(k)} \in \mathbb{R}^{D \times N_k}\right\}_{k=1}^{K}$, $D = d(d+1)/2$

Where column $y_i^{(k)}$ is the aligned vector representation of the matrix $P_i^{(k)}$.

1. For each set $\left\{P_i^{(k)}\right\}_{i=1}^{N_k}$, compute the Riemannian mean $\overline{P}^{(k)}$.
2. Compute $\overline{Q}$, the Riemannian mean of $\left\{\overline{P}^{(k)}\right\}_{k=1}^{K}$.
3. For $k = 1 \ldots, K$:
   (a) For $i = 1, \ldots, N_k$:
      i. Apply PT: $\Gamma_i^{(k)} = \Gamma_{\overline{P}^{(k)} \to \overline{Q}}\left(P_i^{(k)}\right)$
      ii. Project $\Gamma_i^{(k)}$ to a Euclidean space:
      $$z_i^{(k)} = \mathrm{vec}\left(\overline{Q}^{-\frac{1}{2}} \mathrm{Log}_{\overline{Q}}\left(\Gamma_i^{(k)}\right) \overline{Q}^{-\frac{1}{2}}\right)$$



   (b) Collect $Z^{(k)} = \left[z_1^{(k)}, z_2^{(k)}, \ldots, z_{N_k}^{(k)}\right]$
   (c) Apply SVD $Z^{(k)} = U^{(k)} \Lambda^{(k)} V^{(k)}$.
   (d) If $k \neq 1$, for each $j \in \{1, 2, \ldots, D\}$ update the columns of $U^{(k)}$: $u_j^{(k)} \leftarrow \mathrm{sign}\left(\left\langle u_j^{(1)}, u_j^{(k)} \right\rangle\right) u_j^{(k)}$
   (e) Compute $Y^{(k)} = \left(U^{(k)}\right)^T Z^{(k)}$.

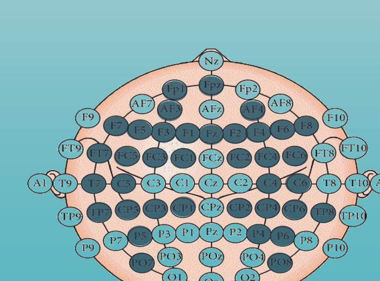
Experimental Results

EEG Data

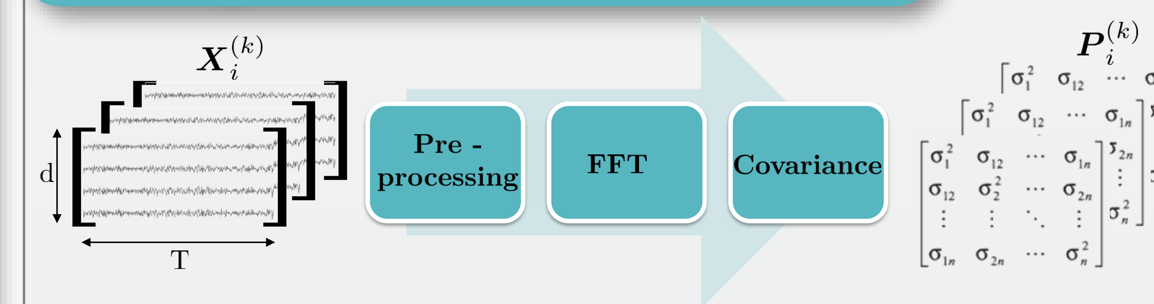
Raw data:
- EEG recordings from 11 subjects of ages ranging from 7 to 16 years.
- 64 EEG electrodes.
- 3 types of stimuli: visual, auditory and somatosensory.

Pre-processing:
- Organizing the recordings into trials of 1 second, and downsampling to 1KHz.
- Excluding of malfunctioning electrodes and highly noisy trials.
- Applying FFT and taking the absolute value.

After pre-processing, the data contain recordings from 37 electrodes, and include 80-500 repeated trials per stimulus for each subject.



Experimental Results



2 Subjects

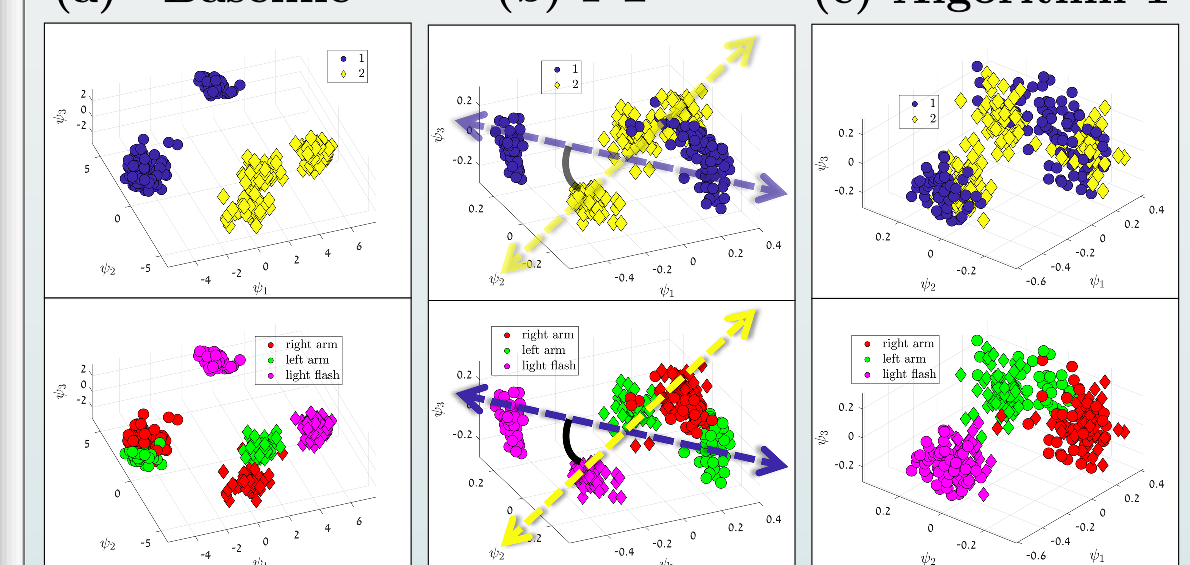


Fig.1. The first three principal components of the representation of the trials obtain by the different algorithms: (a) "Baseline", (b) "PT" and (c) Algorithm 1.

Multiple Subjects

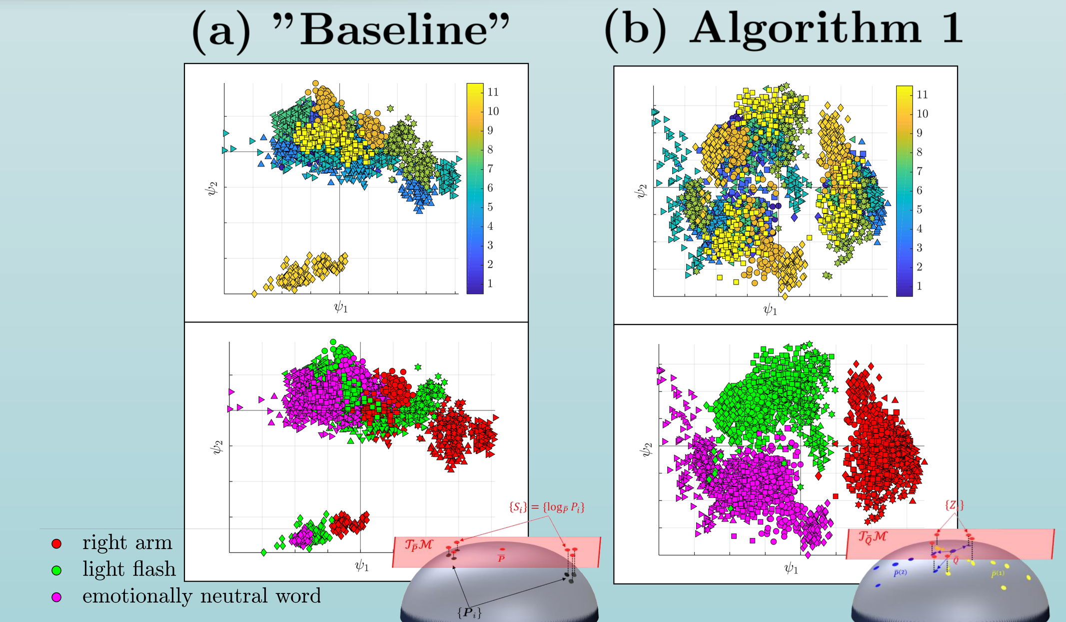
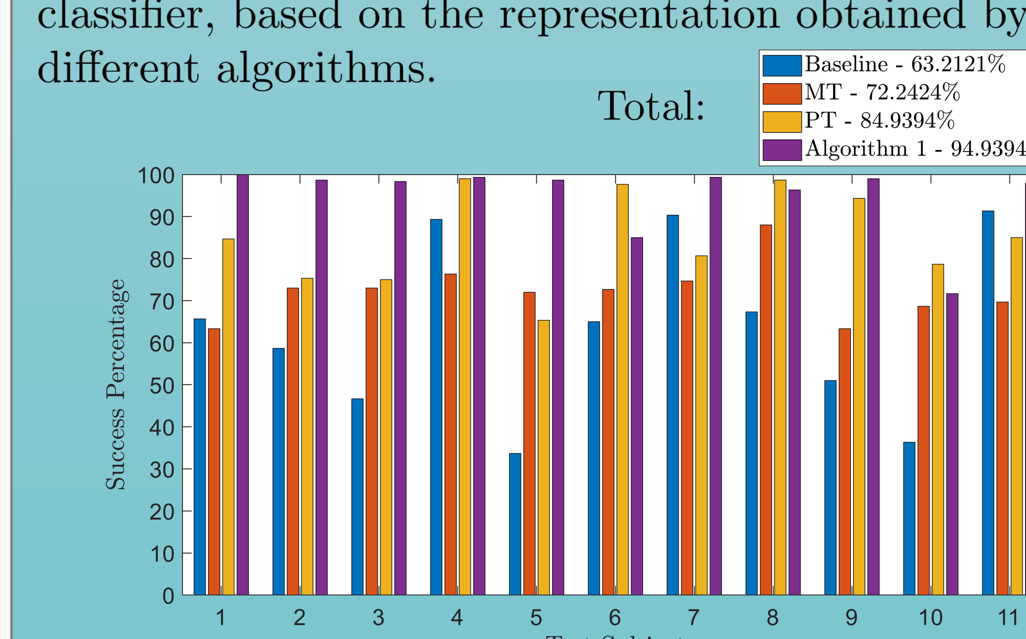


Fig.2. The first two principal components of the representation of the trials obtain by: (a) "Baseline" and (b) Algorithm 1.

Classification Results

Fig.3. Classification results of the responses to three different stimuli from 11 subjects, using a cubic SVM classifier, based on the representation obtained by the different algorithms.



We remark that the sign of the second PC of subjects 5 and 6 was manually flipped (note that their respective angles were 89° and 95°).