SUPER-RESOLUTION DOA ESTIMATION FOR ARBITRARY ARRAY GEOMETRIES USING A SINGLE NOISY SNAPSHOT

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Outline

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DOA Estimation Methods

Classical Methods

- Non adaptive: Conventional delay-sum beamformer (CBF)
- Data adaptive: MVDR, MUSIC, ESPRIT

Compressed Sensing (CS) based Sparse Methods

- On-grid sparse DOA estimation - has *offgrid (discretization) problem*
- Off-grid DOA methods
  - Fixed grid
  - Dynamic grid
- **Gridless method using super-resolution (SR) theory** [Candès and Fernandez-Granda 2014] **for arrays** [Xenaki and Gerstoft 2015]
  - Based on *atomic norm* or *total variation (TV) norm*
  - Uses convex optimization (LMI and SDP)
  - **Only applicable for ULAs** [Xenaki and Gerstoft 2015]

Objective of Proposed Research

- Develop search-free gridless super-resolution DOA method
  - To eliminate offgrid problem of CS
- Extend method to arbitrary array geometries
  - Non-uniform arrays
  - Random Planar 2-D arrays
  - Circular arrays
- Applicable for coherent sources, single snapshot case

Data Model for Arbitrary Array DOA Estimation

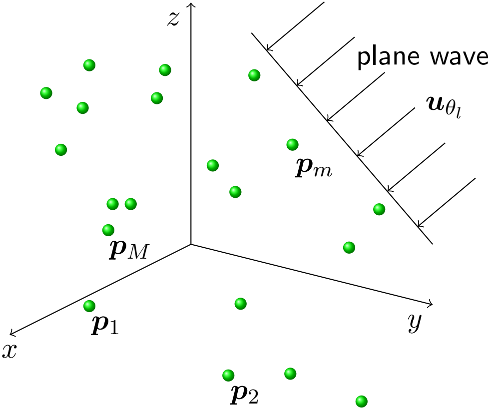
- M sensors, known sensor positions
- L sources with unknown azimuth DOAs $\boldsymbol{\theta} = \{\theta_1, \theta_2 \ldots, \theta_L\}$
- Assumptions:
  - far-field, narrow band sources
  - unknown source amplitudes
  - unknown number of sources (L)
- Objective is to estimate $\boldsymbol{\theta}$, given the data at the sensors
- Array *snapshot vector*

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t) \in \mathbb{C}^{M \times 1}$$

$\boldsymbol{s}(t) = [s_1(t), s_2(t), \ldots, s_L(t)]^{\mathrm{T}} \in \mathbb{C}^{L \times 1}$ is source amplitude vector
$\boldsymbol{n}(t) = [n_1(t), n_2(t), \ldots, n_M(t)]^{\mathrm{T}} \in \mathbb{C}^{M \times 1}$ is noise vector

- *Array manifold* $\boldsymbol{A}(\boldsymbol{\theta}) \triangleq [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_L)] \in \mathbb{C}^{M \times L}$
- *Steering vector* $\boldsymbol{a}(\theta_l)$ for the l-th source from direction θ_l

$$a_m(\theta_l) = e^{-j2\pi f \tau_m(\theta_l)} = e^{-j(2\pi/\lambda)\boldsymbol{u}_{\theta_l}^T \boldsymbol{p}_m}$$



f, λ: frequency, wavelength

$$\tau_m(\theta_l) = \boldsymbol{u}_{\theta_l}^T \boldsymbol{p}_m / v$$

delay at m-th sensor for l-th source

$\boldsymbol{p}_m$: position vector of m-th sensor

$\boldsymbol{u}_{\theta_l}$: unit vector in source direction θ_l

v: wave propagation speed

Sparse DOA Estimation: Discrete vs Continuous

- DOA estimation as a sparse signal reconstruction problem

$$\underbrace{\min_{\mathbf{x}\in\mathbb{C}^K} \|\mathbf{x}\|_1}_{\text{sparse } \mathbf{x}} \quad \text{s.t.} \quad \underbrace{\|\boldsymbol{y} - \boldsymbol{A}(\boldsymbol{\theta}_D)\mathbf{x}\|_2 \leq \epsilon}_{\text{match measurements}}$$

  - $\boldsymbol{A}(\boldsymbol{\theta}_D) \in \mathbb{C}^{M\times K}$: dictionary of steering vectors
  - $\boldsymbol{\theta}_D = \{\theta : \theta = -\pi + 2\pi k/K,\ k = 1, \ldots, K\}$: **discrete** grid of angles

- Sparse DOA estimation over **continuous** domain

$$\min_{x} \|x\|_{\mathcal{A}} \qquad \text{s.t.} \ \ \|\boldsymbol{y} - \mathcal{S}x\|_2 \leq \delta$$

$$x(\theta) = \sum_{l=1}^{L} s_l \delta(\theta - \theta_l), \qquad \mathcal{S}x = \int_{-\pi}^{\pi} a_m(\theta)x(\theta)d\theta, \quad m = 1, \ldots, M$$

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- Source amplitude function in continuous angle domain

$$x(\theta) = \sum_{l=1}^{L} s_l \delta(\theta - \theta_l), \quad \text{with atomic norm } \|x\|_{\mathcal{A}} = \sum_{l=1}^{L} |s_l|$$

- Array snapshot vector

$$\boldsymbol{y} = \mathcal{S}x + \boldsymbol{n}, \quad \text{where } y_m = n_m + \int_{-\pi}^{\pi} a_m(\theta) x(\theta) d\theta, \quad m = 1, \ldots, M$$

$\mathcal{S}(\theta)$ is the array manifold surface with m-th component $a_m(\theta)$

$$a_m(\theta) = e^{-j(2\pi/\lambda)\boldsymbol{u}_\theta^T \boldsymbol{p}_m} = \exp\{-j2\pi(|\boldsymbol{p}_m|/\lambda)\cos(\theta - \angle\boldsymbol{p}_m)\}$$

Proposed Method: Primal and Dual Problems

Primal Problem

$$\min_{x} \|x\|_{\mathcal{A}} \ \text{ s.t. } \|\boldsymbol{y} - \mathcal{S}x\|_2 \leq \delta$$

Dual Problem

$$\max_{\boldsymbol{c}\in\mathbb{C}^M} \Re\{\boldsymbol{c}^H\boldsymbol{y}\} - \delta\|\boldsymbol{c}\|_2 \ \text{ s.t. } \|\mathcal{S}(\theta)^H\boldsymbol{c}\|_\infty \leq 1$$

- $b(\theta) = \mathcal{S}(\theta)^H\boldsymbol{c} = \sum_{m=1}^{M} a_m^*(\theta)c_m$

  $\boldsymbol{c}$ is a vector of Lagrange multipliers (dual variables)
- $|b(\theta)| = 1$ for true source directions

- For ULA, $b(\theta)$ is a polynomial in $z = e^{-j(2\pi/\lambda)d\sin\theta}$

$$\mathcal{S}(\theta)^H\boldsymbol{c} = \sum_{m=1}^{M} c_m e^{-j(m-1)(2\pi/\lambda)d\sin\theta} = \sum_{m=1}^{M} c_m z^{(m-1)}$$

Fourier Domain Polynomial Representation of $b(\theta)$

- For arbitrary arrays, $b(\theta)$ does not have a direct polynomial form
- Fourier Domain approach, motivated by [Rübsamen and Gershman 2009] also [Doron & Doron, 1994]
  - $b(\theta) = \mathcal{S}(\theta)^H \boldsymbol{c} = \sum_{m=1}^{M} a_m^*(\theta)\, c_m$
  - $a_m^*(\theta)$ periodic $\Rightarrow$ $b(\theta)$ periodic $\Longrightarrow$ Fourier Series (FS)
  - $b(\theta) = \sum_{k=-\infty}^{\infty} B_k e^{jk\theta}$, where $B_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\theta) e^{-jk\theta} d\theta$
  - $a_m^*(\theta)$ is smooth, bandlimited $\Longrightarrow$ $b(\theta)$ is bandlimited
    - Finite Fourier Series ($2N+1$ coeffs) $b(\theta) = \sum_{k=-N}^{N} B_k e^{jk\theta}$
  - $b(\theta) \to b(z)\Big|_{z=e^{j\theta}}$ is the *dual polynomial*

Fourier Domain Representation of $a_m(\theta)$

- How to get $\hat{B}_k$'s? $B_k = \sum_{m=1}^{M} \alpha_m[k] \, c_m$
  - $\alpha_m[k]$ are FS coeffs of $a_m^*(\theta) = \exp\{j2\pi(|\boldsymbol{p}_m|/\lambda)\cos(\theta - \angle \boldsymbol{p}_m)\}$
- DFT is used to obtain finite FS of a bandlimited function
  - Compute $\hat{\alpha}_m[k]$ via P-point DFT; $P = 2N+1$, $\Delta\theta = 2\pi/P$

approximation
$\hat{\alpha}_m[k] \approx \alpha_m[k]$

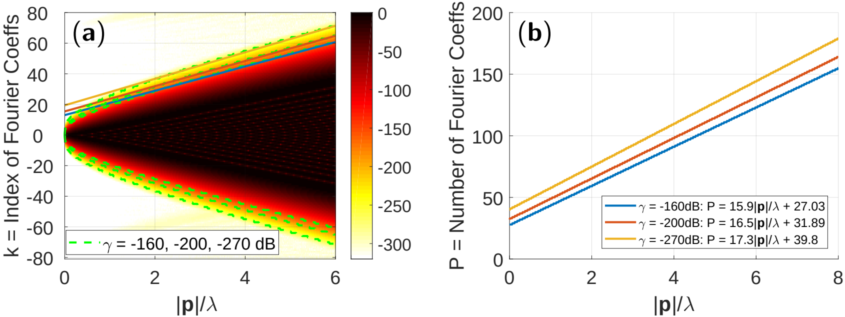
$$\hat{\alpha}_m[k] = \frac{1}{P} \sum_{l=-N}^{N} a_m^*(l\Delta\theta) e^{-j(2\pi/P)lk}$$

- Now, $\hat{B}_k = \sum_{m=1}^{M} \hat{\alpha}_m[k] \, c_m$, so we have $b(\theta) \approx \sum_{k=-N}^{N} \hat{B}_k e^{jk\theta} \rightarrow \hat{b}(z)$,

$$\left[\hat{B}_{-N} \;\; \hat{B}_{-(N-1)} \;\; \ldots \;\; \hat{B}_N\right]^T \triangleq \boldsymbol{h} = \boldsymbol{G}^H \boldsymbol{c},$$

$\boldsymbol{G}^H = \left[\hat{\alpha}_m[k]\right]_{P\times M}$; m-th column has FS coefficients of $a_m^*(\theta)$

- Selection of P for accurate polynomial representation
  - FS bandwidth of $a_m(\theta) = \exp\{-j2\pi(|\boldsymbol{p}_m|/\lambda)\cos(\theta - \angle \boldsymbol{p}_m)\}$
  - Plot magnitude of $\hat{\alpha}_m[k]$ vs. $|\boldsymbol{p}|/\lambda$



(a) DFT spectrum of $a_m^*(\theta)$ ($20\log_{10}|\alpha_k|$ dB) as a function of k and $|\boldsymbol{p}|/\lambda$,
(b) P vs. normalized distance $|\boldsymbol{p}|/\lambda$ for different spectral cutoff levels (γ).

- Linear rule for P w.r.t distance $|\boldsymbol{p}|$ of farthest sensor from reference

For $\gamma = -160$ dB, $P = 15.9|\boldsymbol{p}|/\lambda + 27.03$

Semidefinite Programming and Source Recovery

- Dual Program to Semidefinite Program (SDP)

$$\max_{\boldsymbol{c},\boldsymbol{H}} \Re\{\boldsymbol{c}^H \boldsymbol{y}\} - \delta\|\boldsymbol{c}\|_2; \;\text{ s.t. } \begin{bmatrix} \boldsymbol{H}_{P\times P} & \boldsymbol{G}^H_{P\times M}\boldsymbol{c}_{M\times 1} \\ \boldsymbol{c}^H\boldsymbol{G} & 1 \end{bmatrix} \succeq 0,$$

$$\sum_{i=1}^{P-j} \boldsymbol{H}_{i,i+j} = \begin{cases} 1, & j = 0 \\ 0, & j = 1,\ldots,P-1. \end{cases}$$

SDP has $n = P^2/2 + M$ variables. Worst case complexity $\mathcal{O}(n^3)$

- Recover source DOAs $\hat{\boldsymbol{\theta}}$ from unit-circle roots of nonnegative poly.

$$p(z) = 1 - |\hat{b}(z)|^2 = \sum_{k=-(P-1)}^{P-1} r_k z^k$$

$r_k = \sum_j h_j h^*_{j-k}$ are autocorrelation coeffs of $\boldsymbol{h}_* = \boldsymbol{G}^H\boldsymbol{c}_*$

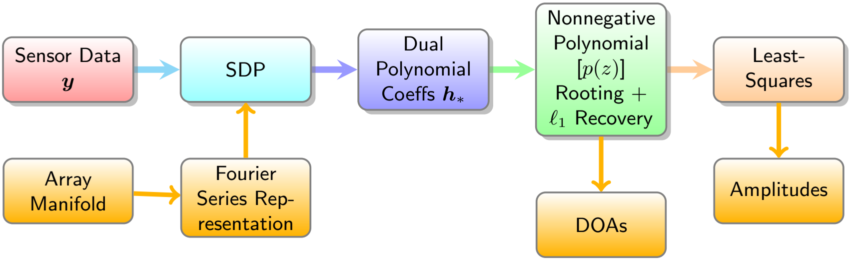
- Recover source amplitudes by least-squares

$$\hat{\boldsymbol{s}} = \boldsymbol{A}(\hat{\boldsymbol{\theta}})^\dagger \boldsymbol{y}$$

Algorithm: Super-Resolution DOA for Arbitrary Array

Input: Array snapshot vector $\boldsymbol{y} \in \mathbb{C}^M$, wavelength λ, number of Fourier coeffs P

1. For the sensor positions, compute $\boldsymbol{G}^H = \left[\hat{\alpha}_m[k]\right]_{P \times M}$ using the DFT to obtain the FS of the array manifold (OFF-LINE)
2. Estimate noise level, and then set δ
3. Using $\boldsymbol{G}^H$ and $\boldsymbol{y}$ as inputs, solve the SDP to find optimal $\boldsymbol{c}_*$
4. Compute the optimal dual polynomial coefficients-vector $\boldsymbol{h}_*$, using $\boldsymbol{h}_* = \boldsymbol{G}^H \boldsymbol{c}_*$
5. Estimate DOAs $\hat{\boldsymbol{\theta}}$ by finding the unit-circle roots of nonnegative polynomial $p(z)$
6. Eliminate extraneous zeros via ℓ_1 recovery
7. Recover source amplitudes $\hat{\boldsymbol{s}}$ by least squares



Running Time Examples

- The observed time complexity seems to grow as P^2
  - SDP has $n = P^2/2 + M$ variables.
- Significantly less than the worst case complexity of $\mathcal{O}(n^3)$

Table: Execution time

Case	P	Radius	Time for SDP	Poly. rooting	# Iterations
1	61	2λ	5.31 sec	0.04 sec	17
2	121	5.87λ	14.79 sec	0.15 sec	18
3	183	9.75λ	57.9 sec	0.37 sec	19

Intel core i7 processor, $M = 40$, Three sources

Outline

1. Introduction and Notation
2. Details of Proposed Method
3. Simulations
4. Conclusion

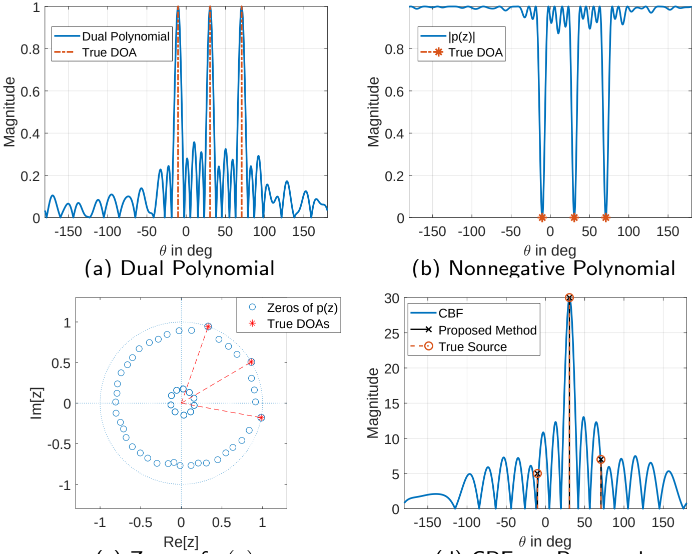
Simulations

- Simulations for Uniform Circular and Random Planar Arrays (Noise-free)
- Performance Evaluation using Success Probability (Noise-free)
- Simulations for Noisy Case
  - White and Colored Noise Examples
  - ℓ_1 Recovery Result
  - Performance Evaluation Vs. Signal to Noise Ratio (SNR)

$$\text{SNR} = \frac{\text{Source Power}}{\text{Noise Power}} \text{ at each sensor}$$

- All Simulations use Coherent Sources and Single Snapshot

Simulation for Uniform Circular Array (UCA)

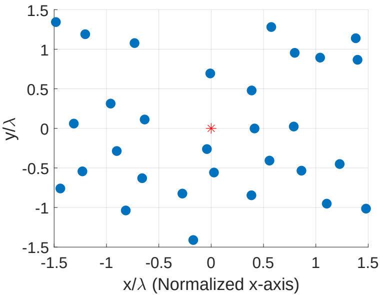


(a) Dual Polynomial (b) Nonnegative Polynomial (c) Zeros of $p(z)$ (d) CBF vs. Proposed

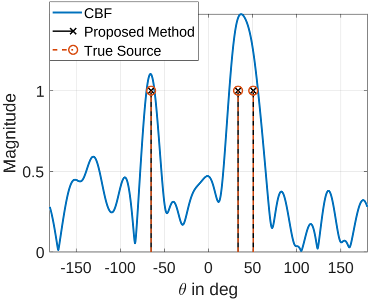
Noise-free case:

Perfect Estimates for DOAs and Mags

UCA with $r = 2\lambda,\ M = 40,\ P = 61.$ Sources at $-10.3°, 30.5°, 70.7°$, magnitudes $5, 30, 7$.



(a) Random Planar Array (RPA)



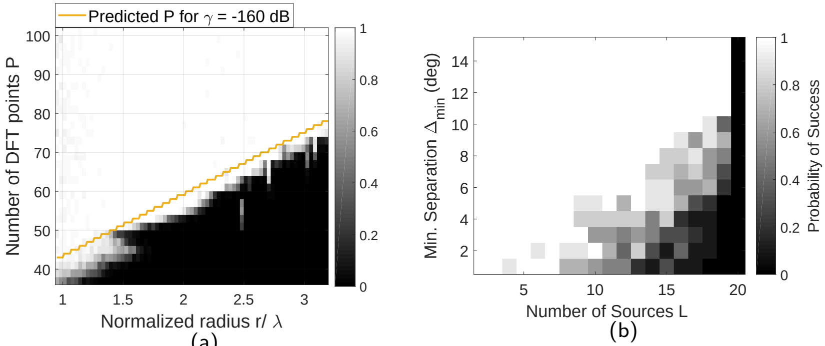
(b) CBF vs. Proposed Method

Result for RPA with $M = 30$, $P = 61$. Farthest sensor at $r/\lambda \approx 2$.

Three sources at DOAs $-65.1°$, $37.5°$, $50.7°$, equal magnitudes.

Noise-free case: estimates of directions and magnitudes are perfect.

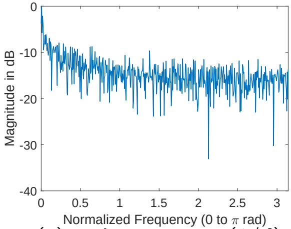
Performance Evaluation for Resolution



Success probability of $M = 40$ UCA (a) versus r/λ and P, with fixed $\Delta_{\min} = 10^\circ$. (b) versus minimum source separation $\Delta_{\min}$ and L with fixed $r/\lambda = 1.59$.

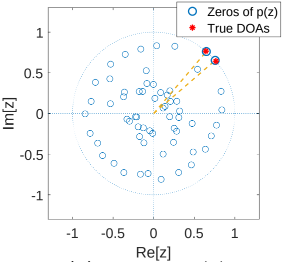
- Success probability
  - Fig. (a): 50 random trials for each P and r/λ. Fixed $\Delta_{\min} = 10^\circ$
    - $L = 10$ sources with random DOAs $\sim \mathcal{U}(-\pi, \pi]$
    - Success declared when all DOAs are estimated within 0.001° error
  - Fig. (b): Fixed radius $r/\lambda = 1.59$, $P = 53$, and 10 trials

Simulations for Noisy Case: Colored Noise Example

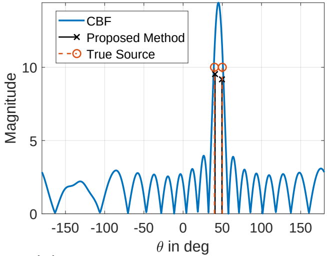


(a) noise spectrum $(1/f)$

DOA RMSE $= 0.6088°$
Amplitude RMSE $= 0.6712$



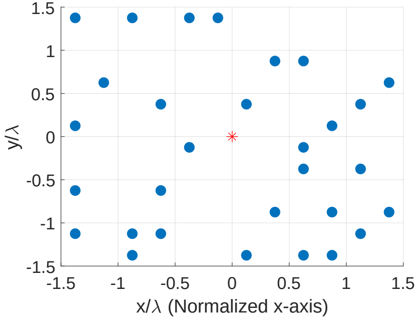
(b) Zeros of $p(z)$



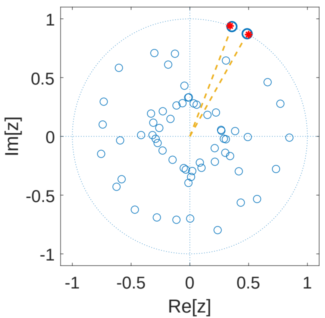
(c) CBF vs. Proposed Method

UCA with $r = 2\lambda$, $M = 40$ sensors, $P = 63$. Two sources at $40°$, $50°$; SNR $= 20\,$dB.

Simulations for Noisy Case: RPA, $M = 30$



(a) Random Planar Array (RPA)



(b) Zeros of $p(z)$

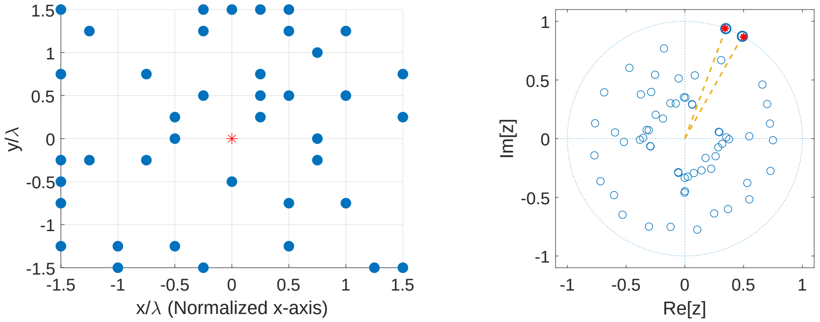
Result for RPA with $M = 30,\ P = 63,\ \max|\boldsymbol{p}| \approx 2\lambda$.

Two equal magnitude sources at 60° and 70°.

SNR $= 20\,$dB. $\delta = 1.4e_n$.

Minimum sensor spacing $= \lambda/4$.

DOA RMSE $= 0.8882^\circ$
Amplitude RMSE $= 0.4693$



(a) Random Planar Array (RPA) (b) Zeros of $p(z)$

Result for RPA with $M = 40,\ P = 63,\ \max|\boldsymbol{p}| \approx 2\lambda$.

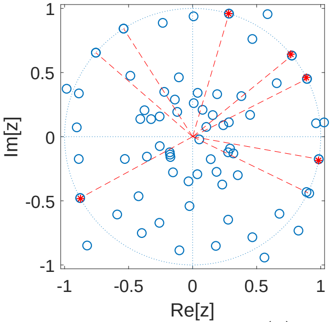
Two equal magnitude sources at 60° and 70°.

SNR $= 20\,$dB. $\delta = 1.4e_n$.

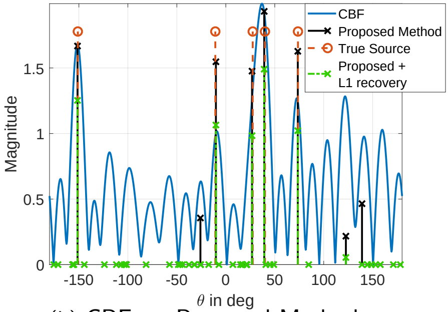
Minimum sensor spacing $= \lambda/4$.

DOA RMSE $= 0.5583^\circ$
Amplitude RMSE $= 0.3652$

Simulations for Noisy Case: ℓ_1 Recovery



(a) Zeros of $p(z)$



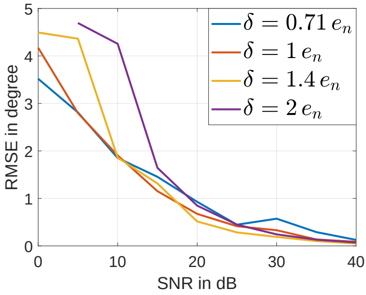
(b) CBF vs. Proposed Method

Result for UCA with radius $= 2\lambda$, $M = 40$, $P = 63$. $\delta = 1.4e_n$.

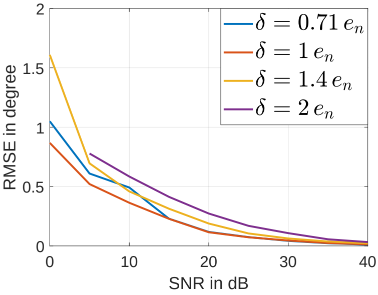
Five sources with SNR $= 5\,$dB at $-10.7^\circ, 27.5^\circ, 40^\circ, 73.7^\circ$ and -151.1°

DOA RMSE $= 0.5617^\circ$
Amplitude RMSE $= 0.2016$

- Extraneous roots from polynomial rooting
  - Need ℓ_1 recovery to remove unwanted roots
- Estimate amplitudes by least-squares



(a) Source Separation $= 10^\circ$



(b) Source Separation$= 30^\circ$

DOA accuracy vs. SNR for UCA with $r = 2\lambda$, $M = 30$, and $P = 63$.

50 trials, two sources at random DOAs in each trial.

Additive noise $\mathcal{CN}(0, \sigma)$ per sensor $\Rightarrow e_n = \mathbb{E}[\|\boldsymbol{n}\|_2] = \sqrt{M\sigma^2}$

Outline

Conclusion

- Search-free gridless SR DOA method for arbitrary arrays using single noisy snapshot
  - Formulated problem as an atomic norm minimization
  - Fourier domain approach for polynomial representation of manifold
  - Finite SDP formulation for arbitrary arrays, solvable in polynomial time
- No strong source masking weak source problem, unlike CBF
- Applicable for coherent sources, single snapshot, and colored or white noise scenarios
- Larger impact: Applicable to generic data model involving periodic measurement functions, and to other applications.

Thank You!