Disjunct Matrices for Compressed Sensing

P. Sasmal, S. S. Thoota, **C. R. Murthy**

Dept. of Electrical Communication Engineering (ECE)
Indian Institute of Science, Bangalore, India
email:{pradipsasmal, thoota, cmurthy}@iisc.ac.in

ICASSP 2019
May 11, 2019

Outline

Motivation

- Basic pursuit (BP) & orthogonal matching pursuit (OMP): polynomial complexity in problem dimension
  - Impractical and expensive in high dimensional settings
- Verifying conditions based on spark and RIP is not easy
  - Hence, in practice, it remains unknown whether a given instantiation of the measurement matrix satisfies these properties

Goal

Identify a property of a matrix that is easy to verify and also supports low computational complexity sparse recovery algorithms, while perhaps requiring a larger number of measurements for success.

Contributions

- We connect non-adaptive group testing and compressed sensing
  - Disjunctness property of binary matrices is also very useful in recovering sparse signals
- We exploit the disjunctness property to present an ultra-low complexity algorithm for identifying the support as well as recover the nonzero coefficients of the sparse signal
  - Non-iterative algorithm, very fast
- We extend the disjunctness property of a binary matrix to sparse matrices. We show that a similar non-iterative and fast sparse recovery algorithm is possible

Notation

- The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.
- The i-th entry of x is denoted by x_i.
- $\Phi(:, i)$ and $\Phi(j, :)$ denote the i-th column and j-th row of Φ, respectively, and $\Phi(j, i)$ denotes the (j, i)th entry of Φ.
- The support of x is $\{i : x_i \neq 0\}$, denoted by $\mathrm{supp}(x)$.
- Let $S \subset [n]$, then $x_S \triangleq (x_i)_{i \in S}$ and $\Phi_S \triangleq (\Phi(:, i))_{i \in S}$

Disjunct Matrix

Definition 1

An $m \times M$ binary matrix Φ is called t-disjunct if the support of any column is not contained in the union of the supports of any other t columns.

Implications:

- If we take a submatrix Φ_S with $|S| = t + 1$, then for $i \in [t+1]$, there exists j_i such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [t+1] \setminus i$
- This observation will be crucial for non-iterative recovery of almost all sparse signals

C. L. Chan, S. Jaggi, V. Saligrama and S. Agnihotri, "Non-Adaptive Group Testing: Explicit Bounds and Novel Algorithms," in IEEE Transactions on Information Theory, vol. 60, no. 5, pp. 3019-3035, May 2014.

t^e-disjunct Matrix

Definition 2

A matrix Φ is t^e-disjunct if, given any $t+1$ columns of Φ with one designated column, there are $e+1$ rows with a 1 in the designated column and a 0 in each of the other t columns.

Implications:

- If we take a submatrix Φ_S with $|S| = t+1$, then for $i \in [t+1]$, there exists $j_i^1, \ldots, j_i^{e+1}$ such that $\Phi_S(j_i^d, i) = 1$ and $\Phi_S(j_i^d, l) = 0$ for all $l \in [t+1] \setminus i$ and $d = 1, \ldots, e+1$.
- We exploit this property for recovering all signals with a given max. sparsity level

A. J. Macula " Error-correcting nonadaptive group testing with d^e-disjunct matrices," Discrete Applied Mathematics, Vol 80, Issues 23, Pp 217-222 11 December 1997.

Disjuntness of a Constant Column Weight Binary Matrix

Theorem 1

Let Φ be a $m \times M$ matrix with each column containing q ones and the overlap (i.e., the size of the intersection of the supports) between any two distinct columns is at most r. Then Φ is $\lfloor \frac{q-1}{r} \rfloor$-disjunct.

A. Mazumdar, "Nonadaptive Group Testing With Random Set of Defectives," IEEE Transactions on Information Theory, vol. 62, no. 12, pp. 7522-7531, Dec. 2016.

Relation with Spark

Definition 3

The spark of a matrix is the smallest number of linearly dependent columns in the matrix

- Necessary and sufficient condition for uniqueness
  - If spark(Φ) $= k$, sparse vectors with up to $k/2$ nonzero entries (and no more) can be uniquely recovered from $y = \Phi x$

Theorem 2

The spark of a t-disjunct matrix is at least $t + 1$

Proof 1

Follows from the definition of a disjunct matrix.

Relation with Mutual Coherence

Definition 4

The mutual coherence μ_Φ of Φ is the maximum absolute inner product between any two distinct normalized columns of Φ

Theorem 3

A matrix Φ containing the same number of ones in each column is $(\lfloor \mu_\Phi^{-1} \rfloor - 1)$-disjunct.

Proof 2

Follows from the fact that if each column of Φ contains q ones and the overlap between any two columns is at most r, then its mutual coherence $\mu_\Phi \leq \frac{r}{q}$.

Recovery of all sparse signals using Binary Matrix

- Suppose $\Phi(:, i)$ contains q_i ones for $i \in [M]$, $q_{\min} \triangleq \min\{q_1, \ldots, q_M\}$, and that the overlap between any two distinct columns is at most $r_{\max}$

Theorem 4

Φ is t^e-disjunct for any $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e + 1 \geq q_{\min} - t r_{\max}$

Theorem 5

Let Φ be a binary matrix with every column containing at least $q_{\min}$ ones and with the overlap between any two distinct columns at most $r_{\max}$. Then any $\lfloor \frac{q_{\min}}{2r_{\max}} \rfloor$-sparse vector can be uniquely recovered from $y = \Phi x$

Proof (and a fast recovery algorithm)

Support recovery

$S = \{j : |supp(\Phi(:, i)) \cap supp(y)| > \frac{q_{\min}}{2}\}$ is the support of x.

Non-zero coefficient recovery

Step-1 : As Φ is t^e-disjunct for some $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e \geq q_{\min} - tr_{\max} - 1$, it is also $\lfloor \frac{q_{\min}}{2r_{\max}} \rfloor^{\frac{q_{\min}}{2}}$-disjunct.
Step 2 : As a result, whenever $s \in S$, for $\Phi_S(:, s)$ there exist $j_s^1, \ldots, j_s^{e+1}$ rows such that $\Phi_S(j_s^d, s) = 1$ and $\Phi_S(j_s^d, l) = 0$ for $l \in S \setminus s$ and $d = 1, \ldots, e+1$.
Step 3 : Thus, we can directly recover

$$x_s = \begin{cases} y_{j_s^d},\ d = 1, \ldots, e+1 & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Recovery of Almost All Sparse Signals Using a t-Disjunct Binary Matrix

- **Assumption :** $y_j = \sum_{l \in supp(\Phi_S(j,:))} x_l \neq 0, \ \forall\, j \in [m]$
- This holds (a) with probability 1 if x is drawn from a generic random model; and (or?) (b) x is a non negative sparse signal

Support recovery

$S = [M] \setminus \bigcup_{j:y_j=0} supp(\Phi(j,:))$

Non-zero coefficient recovery

Step-1 : As Φ is t-disjunct, for $i \in [k]$, there exists j_i such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [k] \setminus i$
Step 2 : Set

$$x_i = \begin{cases} y_{j_i}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Disjunctness of a Sparse Matrix

Definition 5

An $m \times M$ sparse matrix Φ is said to be t-disjunct if the support of any column is not contained in the union of the supports of any t other columns

- Let Φ be a sparse matrix where $\Phi(:, i)$ contains q_i non-zeros for $i \in [M]$ with $q_{\min} \triangleq \min\{q_1, \ldots, q_M\}$
- Let the cardinality of the intersection between support of any two distinct columns be at most $r_{\max}$

Theorem 6

Φ is t^e-disjunct if $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e + 1 \geq q_{\min} - t r_{\max}$.

Recovery of All Sparse Signals Using a Sparse Matrix

- Consider the linear system $y = \Phi x$, where $k < \frac{q_{\min}}{2 r_{\max}}$

Support recovery

$S = \{j : |supp(\Phi(:, i)) \cap supp(y)| > \frac{q_{\min}}{2}\}$.

Non-zero coefficient recovery

(**Step 1**) and (**Step 2**): same as the binary case
Step 3: Set

$$x_s = \begin{cases} \frac{y_{j_s^d}}{\Phi_S(j_s^d, s)}, \; d = 1, \ldots, e+1 & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Recovery of almost all sparse signals using Sparse matrix

- **Assumption :** $y_j = \sum_{l \in supp(\Phi_S(j,:))} \Phi(j,l) x_l \neq 0, \ \forall \ j \in [m]$.
- This holds for same conditions as given for binary matrices.
- Consider the linear system $y = \Phi x$, where Φ is $t-$disjunct and $k < t+1$.

Support recovery

$S = \{i : supp(\Phi(:,i)) \subseteq supp(y)\}$

Non-zero coefficient recovery

Step 1: same as in binary case.
Step 2: Now set

$$x_i = \begin{cases} \frac{y_{j_i}}{\Phi(j_i,i)}, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Simulation Results

- We use the binary sensing matrix Φ of size $q^2 \times q^{r+1}$ constructed by Devore[4] for q being prime power and $r > 1$.
- Every column of Φ has q ones and the overlap between any two distinct columns is at most r
- Φ is $\lfloor \frac{q-1}{r} \rfloor$-disjunct and t^e-disjunct with $t < \lfloor \frac{q}{r} \rfloor$ and $e + 1 \geq q - tr$
- As an example, we take Φ of size $(29)^2 \times (29)^3$ Therefore, Φ is $14-$disjunct and also 7^{14}-disjunct (i.e., $t = 7, e = 14$) and $\mu_\Phi \leq \frac{2}{29}$

[4] R.A. DeVore, "Deterministic constructions of compressed sensing matrices," Volume 23, Issues 46, Pp 918-925, 2007.

Continued ...

- We consider sparsity $k \leq 33$. For each k, we generate 1000 random $k-$sparse vectors
- Our algorithm recovers sparse vectors with $k = 15$ in all 1000 trials, as expected
- Further, the algorithm can recover x with much higher sparsity, up to $k = 33$, in all 1000 trials
- An existing non-iterative sparse recovery algorithm[5] can recover the unknown sparse vector x only up to sparsity 7 exactly in all 1000 trials. Beyond $k = 9$, it fails to recover even a single unknown sparse vector
- This is because the existing algorithm requires $4k < q$, i.e., $k < 8$, in order to ensure that each nonzero entry in x occurs at least $q/2$ times in y.

[5] M. Lotfi and M. Vidyasagar, "A Fast Noniterative Algorithm for Compressive Sensing Using Binary Measurement Matrices," in IEEE Transactions on Signal Processing, vol. 66, no. 15, pp. 4079-4089, 1 Aug.1, 2018.

Run-time Comparison

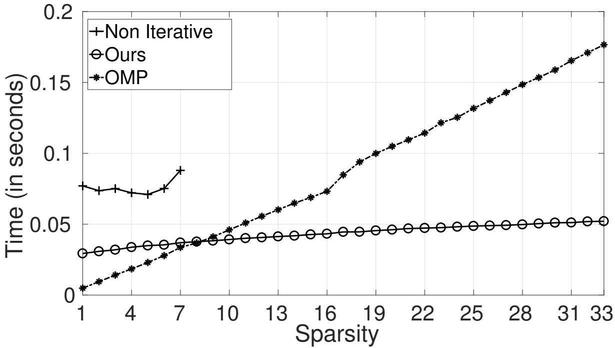

Figure: Avarage runtime comparison between Our proposed method, OMP and the existing non-iterative algorithm for matrix size $(29)^2 \times (29)^3$

Future Work

- Deriving bounds on the number of rows required for the measurement matrix to satisfy t-disjunctness
- Sparse signal recovery guarantees for disjunct matrices in noisy measurement settings.

THANK YOU!