Solving Continuous-Domain Inverse Problems Exactly with B-splines

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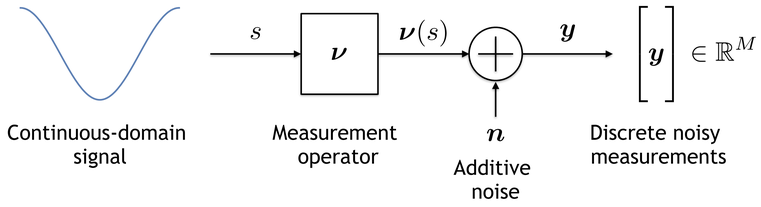
Introduction

- **Model**: Finite measurements of continuous-domain 1D signals



- **Goal**: Reconstruct signal s based on measurements $\boldsymbol{y}$ by solving variational inverse problem
- **Assumption**: s is **sparse** in a transform domain (e.g. TV regularization)
- **Contributions**:
  - ➡ Discretization of continuous-domain inverse problem
  - ➡ Design of a multiresolution algorithm
- **Key features**:
  - ➡ General framework (choice of measurements and regularization)
  - ➡ **Exact** discretization in the continuous domain
  - ➡ **Well-conditioned** optimization task → effective algorithms
  - ➡ **Multiresolution** approach → scalability

Continuous-Domain Inverse Problem

- **Problem formulation**:

$$\arg\min_{f} \|\boldsymbol{\nu}(f) - \boldsymbol{y}\|_2^2 + \lambda \|\mathrm{D}^{N_0}\{f\}\|_{\mathcal{M}} \qquad (1)$$

- λ : regularization parameter
- D^{N_0} : N_0-th order derivative → **generalized TV regularization**
- $\|\cdot\|_{\mathcal{M}}$: sparsity-promoting norm

- **Representer theorem** *[Unser et al. 2017]*: (1) has **spline** solutions s

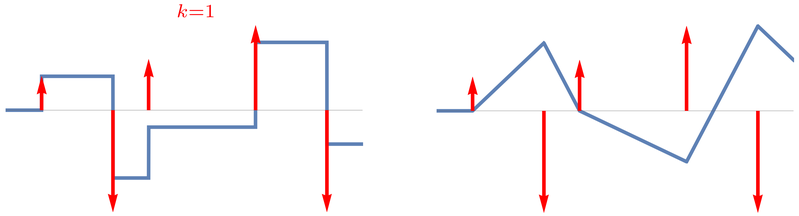
$$\mathrm{D}^{N_0}\{s\}(x) = \sum_{k=1}^{K} a_k \delta(x - x_k) \qquad K \leq M - N_0$$

  - K : sparsity index
  - M : number of measurements

➡ **Sparse solutions (few jumps)!**

- **Splines**: piecewise polynomials

$\sum_{k=1}^{K} a_k \delta(x - x_k)$ = innovation of the spline (jumps)



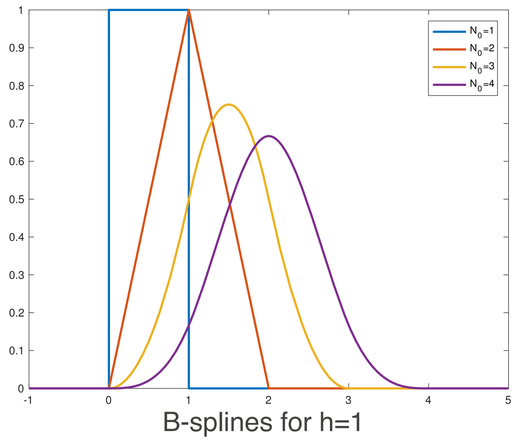
B-spline-based Discretization

- **Basis function**: scaled B-splines $\beta_{N_0,h}$

➡ **Compact support!**

- **Search space**: splines on a uniform grid with knot spacing h

$$\left\{ \sum_{k\in\mathbb{Z}} c[k] \beta_{N_0,h}(x - kh) \right\}$$



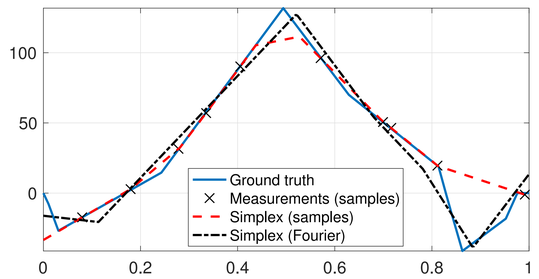
- **Discretized problem:**

$$\arg\min_{\boldsymbol{c}\in\mathbb{R}^N} \|\mathbf{H}\boldsymbol{c} - \boldsymbol{y}\|_2^2 + \lambda \|\mathbf{L}\boldsymbol{c}\|_1 \qquad (2)$$

➡ **Exact discretization!**
➡ **Standard discrete convex problem with ℓ_1 regularization**
➡ **Good conditioning (compact support of B-splines)**

Experimental Results

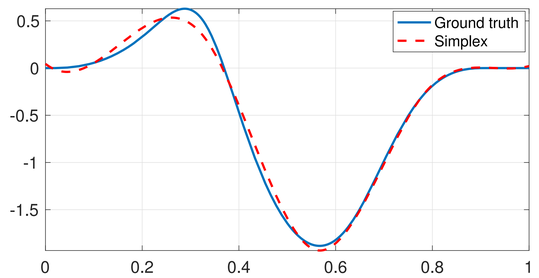
- **N_0 = 2, M = 10 measurements**



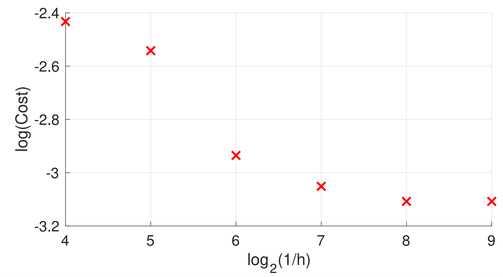
Reconstruction results with grid size h = $1/2^8$

Measurements: samples and Fourier-domain samples

- **N_0 = 4, M = 10 Fourier-domain samples**



Reconstruction result with grid size h = $1/2^9$



Evolution of the optimal cost of (2) as the grid gets finer

References

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