Geometric Invariants for Sparse Unknown View Tomography

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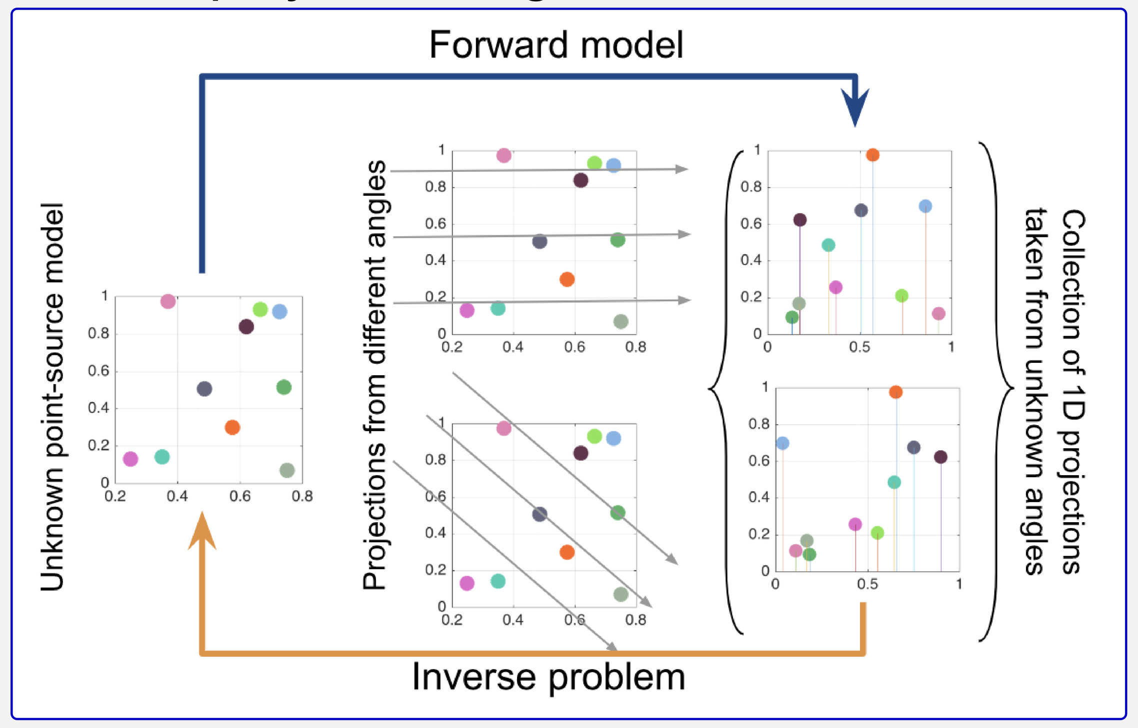
Problem statement

Recover a 2D point-source signal from projection lines taken from unknown and random projection angles.



$$s_\ell[u] = \mathcal{D}\{\mathcal{P}_{\theta_\ell} I\}[u] + \varepsilon_\ell[u], \quad \ell \in \{1, \ldots, L\}$$

$$I(x, y) = \sum_{k=1}^{K} \delta(x - x_k, y - y_k)$$

- $\{s_\ell\}_{\ell=1}^{L}$: L observed projection lines.
- I: Unknown model with K point sources located at $\{(x_k, y_k)\}_{k=1}^{K}$.
- $\mathcal{P}_\theta$: 1D projection operator along θ direction.
- θ: the angle between the projected direction and the horizontal x axis. $\theta \sim [0, 2\pi)$.
- $\mathcal{D}$: Discretization operator
- ε: the additive Gaussian noise, $\varepsilon_\ell \sim \mathcal{N}(0, \sigma^2)$.

Given $\{s_\ell\}_{\ell=1}^{L}$, find the locations of the point sources , i.e. $\{(x_k, y_k)\}_{k=1}^{K}$!

Method

Recover the signal without recovering the projection angles by the use of rotational invariant features.

(1) Estimate rotation-invariant features

(2) Extract geometry info of the model

(3) Point source recovery

(1) Rotation invariant features

Analytical expression of the features:

$\hat{g}_\theta = DFT\{\mathcal{D}\mathcal{P}_\theta I\}$

$$\mu[\nu] = \mathbb{E}_\theta\{\hat{g}_\theta[\nu]\} \approx \sum_{k=1}^{K} J_0\left(\frac{\pi r_k}{R}\nu\right)$$

$$C[\nu] = \left(\mathbb{E}_\theta\{|\hat{g}_\theta[\nu]|^2\} - K\right)/2.$$

$$\approx \sum_{m=1}^{K}\sum_{n=m+1}^{K} J_0\left(\frac{\pi d_{m,n}}{R}\nu\right)$$

Radial distance $r_k = \sqrt{x_k^2 + y_k^2}$

Pairwise distance $d_{m,n} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}$

J_0: the Bessel function of the first kind and the zeroth order.

R: the compact support of the point sources.

Empirical estimation of the features:

$\hat{s}_\ell = DFT\{s_\ell\}$

$$\hat{\mu}[\nu] = \frac{1}{L}\sum_{\ell=1}^{L} \hat{s}_\ell[\nu],$$

$$\hat{C}[\nu] = \left(\frac{1}{L}\sum_{\ell=1}^{L} |\hat{s}_\ell[\nu]|^2 - (2M+1)\sigma^2 - K\right)/2$$

M: # of quantization bins

(2) Extracting geometry information

Use asymptotic behavior of Bessel functions.

$$J_0(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right), \; z \gg 1/4$$

Approximate the features μ and C with sums of complex exponentials.

$$\hat{\mu}[\nu] \approx \sum_{k=1}^{K} \frac{e^{\iota(\pi\nu r_k/R - \pi/4)} + e^{-\iota(\pi\nu r_k/R - \pi/4)}}{\sqrt{2\pi^2 r_k/R\nu}},$$

Apply Prony or MUSIC in order to extract $\{r_k\}_{k=1}^{K}$ and $\{d_{m,n}\}_{m,n=1}^{K}$.

(3) Recover point source model from distance distributions

Approximate radial and pairwise distance distributions from the rotation invariant features. Use the method in [1] in order to recover the location of the point sources.

Numerical Results

- Generate the coordinates of K points randomly on $[-1, 1] \times [-1, 1]$.
- $L = 10^4$ projections of the point source model with projection angles uniformly sampled from $[0, 2\pi)$.
- Generate 100 random point source realizations.
- Use earth mover's distance to quantify the performance and the success-rate recovery.

Extracting the radial and pairwise distances

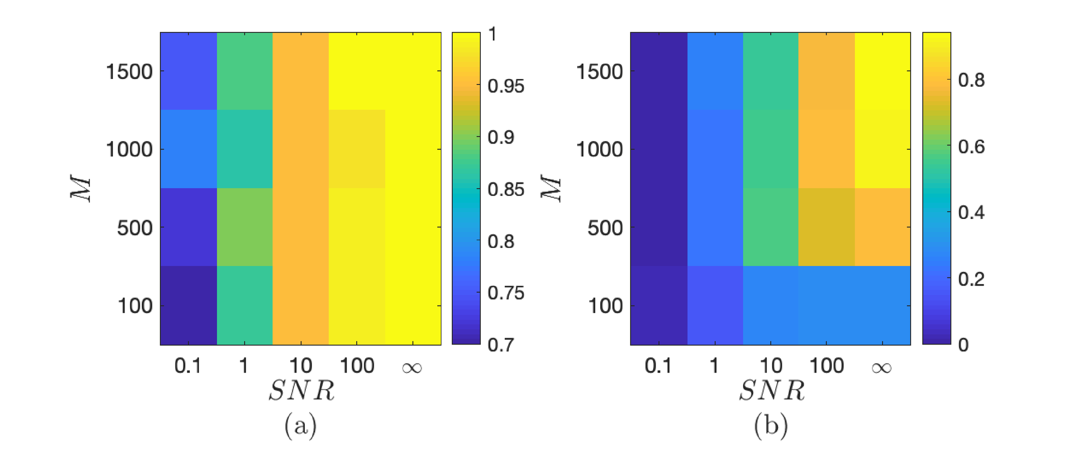


Figure: Successful recovery rate for (a) the radial distance distribution, (b) the pairwise distance distribution.

- SNR increases, the success recovery rate improves. Note that $SNR = \infty$ refers to the no noise regime.
- Comparing (a) and (b) shows that success rate for recovering the pairwise distances is lower compared to the radial distances.

Point source recovery

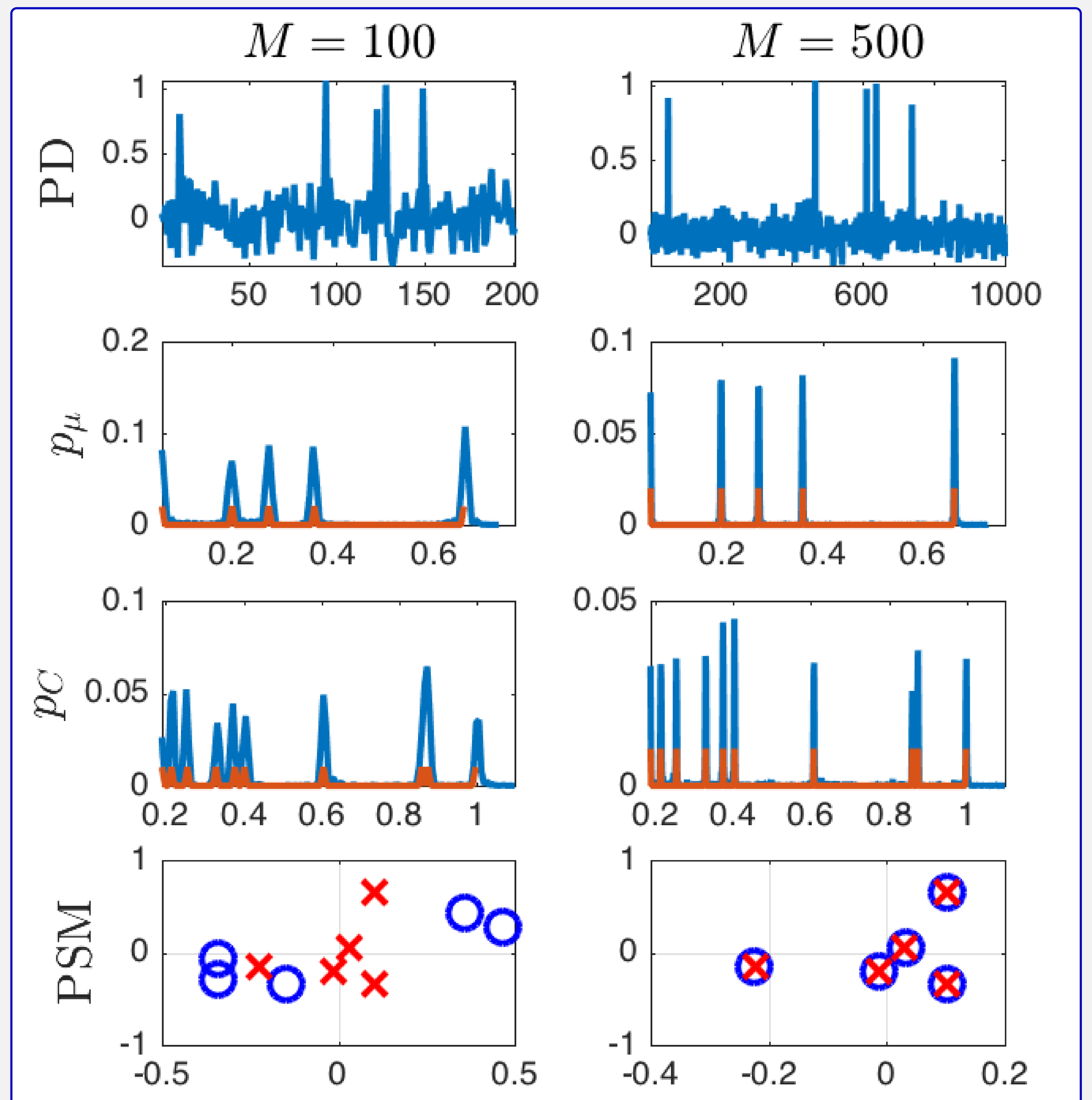


Figure: PD (the first row):projection data for $\mathrm{SNR} = 1$, p_μ, p_C: the estimated distribution of the radial and pairwise distances, a comparison of the true and the recovered point source models (PSM, the last row). The blue circles and red crosses mark the recovered and true point source models.

Conclusion

- A pipeline to recover a point source model from a set of projections taken from unknown angles.
- Instead of first recovering the angles and then solving a tomography problem, we directly recover the locations of the point sources using a set of rotational invariant features that are estimated from the projection data.
- Used the rotation invariant features to, 1) extract the geometry information of the model, 2) extract radial and pairwise distance distributions to recover the locations of the point sources.

References

[1] Shuai Huang and Ivan Dokmanic. Reconstructing point sets from distance distributions. *CoRR*, abs/1804.02465, 2018.