Image Reconstruction by Orthogonal Moments Derived by the Parity of Polynomials

Bo Yang, Wei Tang
and Xiaofeng Chen

Northwestern Polytechnical University

Outline

- Introduction
- Design of Orthogonal Moments (OGMs)
- Hermite-Fourier Moments (HFMs)
- Experiments
  - Image reconstruction
  - Moment invariant recognition
- Conclusions

OGMs

- Based on orthogonal polynomials and functions
  - Useful feature descriptors
  - Advantages in computation and numerical stability
  - Legendre, Hermite, Chebyshev, Zernike polynomials etc.

- OGMs

$$v_{mn} = c_m c_n \iint_{\Omega} P_m(x) P_n(y) f(x, y) dxdy$$

$$m, n = 0, 1, 2, \cdots$$

- Orthogonality

$$\int_{\Omega_1} P_m(x) P_n(x) w(x) dx = h_n \delta_{mn}$$

$w(x)$ is weight function; δ_{mn} is ***Kronecker*** delta.

Approach to define OGM

- Use existing OG polynomials
  - Legendre, Gaussian-Hermite, Krawtchouk moments …
- Seek functions that meet orthogonality
  - Exponent, Bessel moments …
- A common way to define OGMs in polar coordinates
  - Jacobi-Fourier, Zernike, Fourier-Mellin, …

$$\Psi_{nl}(r,\theta) = \boxed{R_n(r)} \cdot \exp(il\theta)$$

⇩

$$\int_0^{1/+\infty} R_m(r) R_n(r) w(r) r dr = d_n \delta_{mn}$$

Orthogonality should be over $[0,1]$ or $[0,+\infty)$.

New Orthogonal Radial polynomials

- New $R_m(r)$ with $r \in [0,1]$ or $[0,+\infty)$ can generate OGM
- $R_m(r)$ can be generated by parity of orthogonal polynomials
  - Hermite, Legendre polynomials have obvious parity
- Hermite polynomials
  - Definition

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n} \exp\left(-x^2\right)$$

  - Orthogonality

Orthogonality is over $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) \exp\left(-x^2\right) dx = \sqrt{\pi}\, 2^n n! \delta_{mn}$$

Normalized Hermite Polynomials (NHPs)

- NHPs
  - Formation

$$\hat{H}_n(x;\sigma)=\frac{1}{\sqrt{2^n n!\sqrt{\pi}\sigma}}H_n\left(\frac{x}{\sigma}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

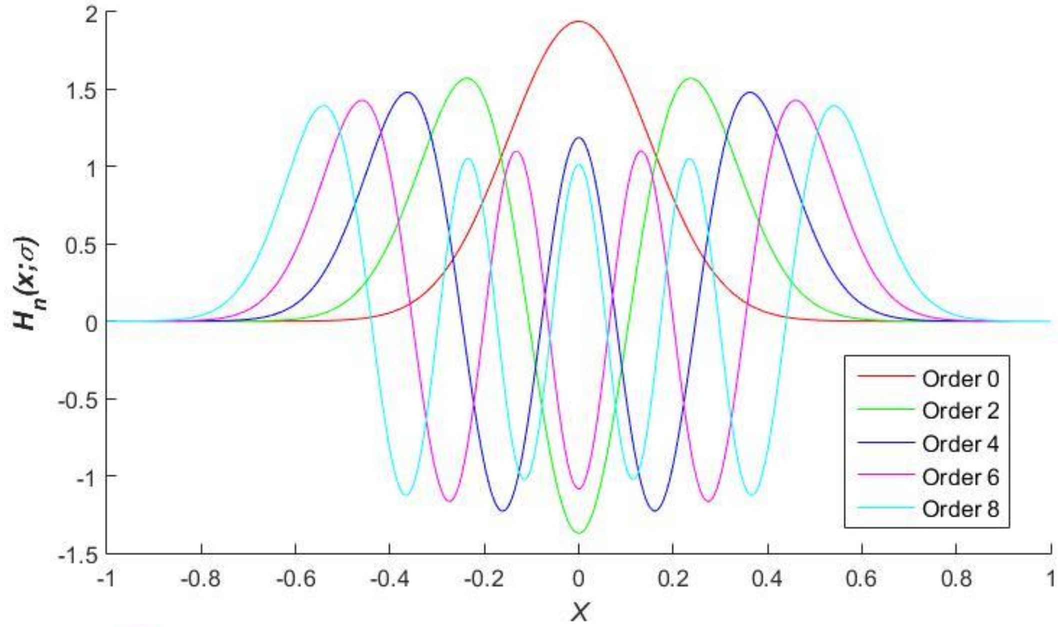
  - Orthogonality

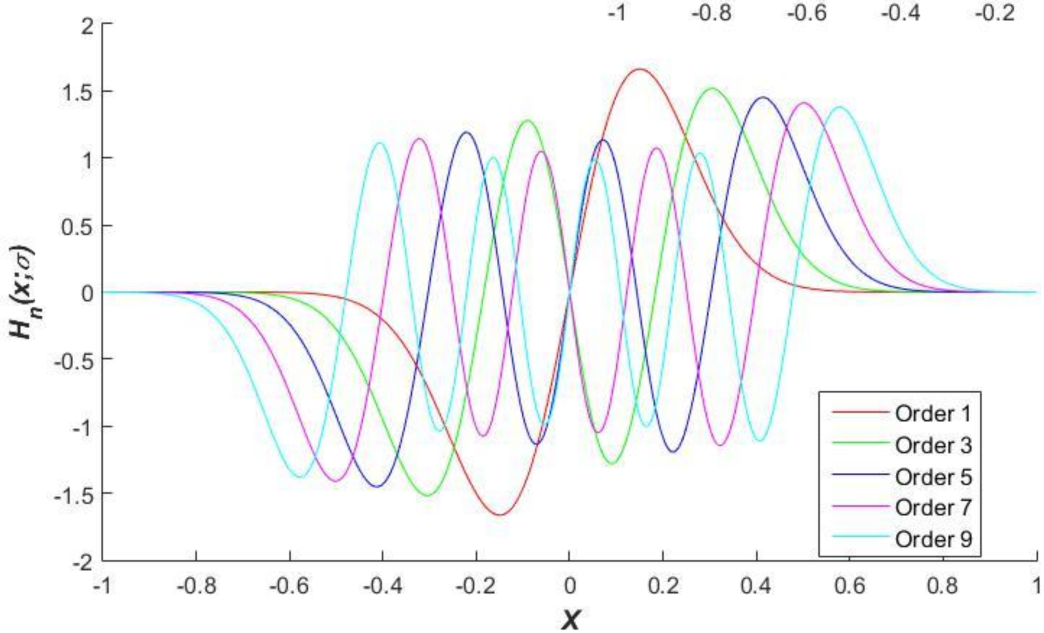
$$\int_{-\infty}^{\infty}\hat{H}_m(x;\sigma)\hat{H}_n(x;\sigma)dx=\delta_{mn}$$

  - **Parity**

$$\begin{cases}\hat{H}_n(-x;\sigma)=\hat{H}_n(x;\sigma) & \text{for } n \text{ is even}\\ \hat{H}_n(-x;\sigma)=-\hat{H}_n(x;\sigma) & \text{for } n \text{ is odd}\end{cases}$$

Plotting of NHPs





$$\sigma = 0.15$$
$$x \in [-1,1]$$

Radial Hermite Polynomials (RHPs)

- Orthogonality from Parity
  - Formulation of RHP

$$\tilde{H}_n(r;\sigma) = \frac{1}{\sigma\sqrt{2^{n-1}n!\sqrt{\pi}}}\left(\frac{\sigma}{r}\right)^{\frac{1}{2}} H_n\left(\frac{r}{\sigma}\right)\exp\left(-\frac{r^2}{2\sigma^2}\right)$$

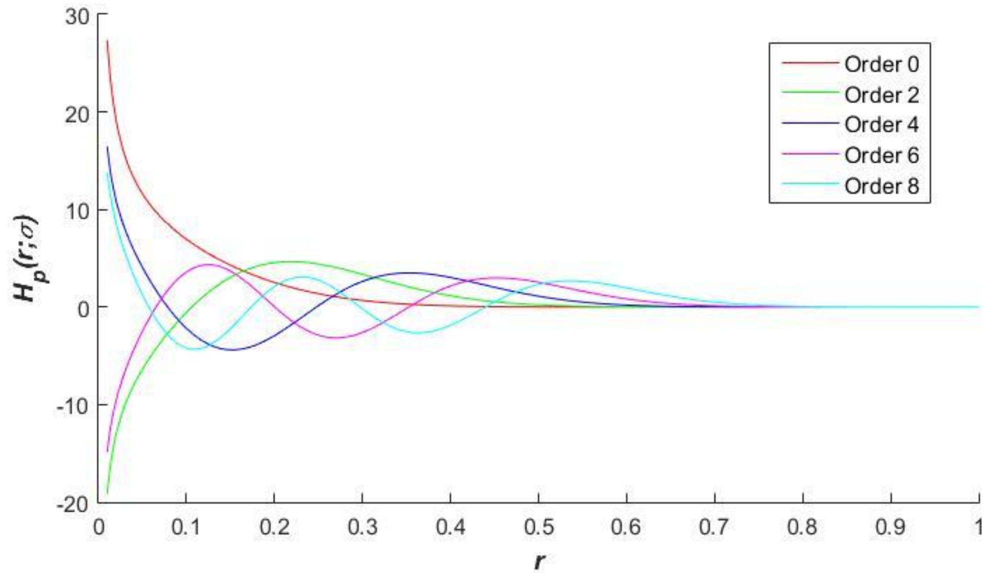
  - Orthogonality

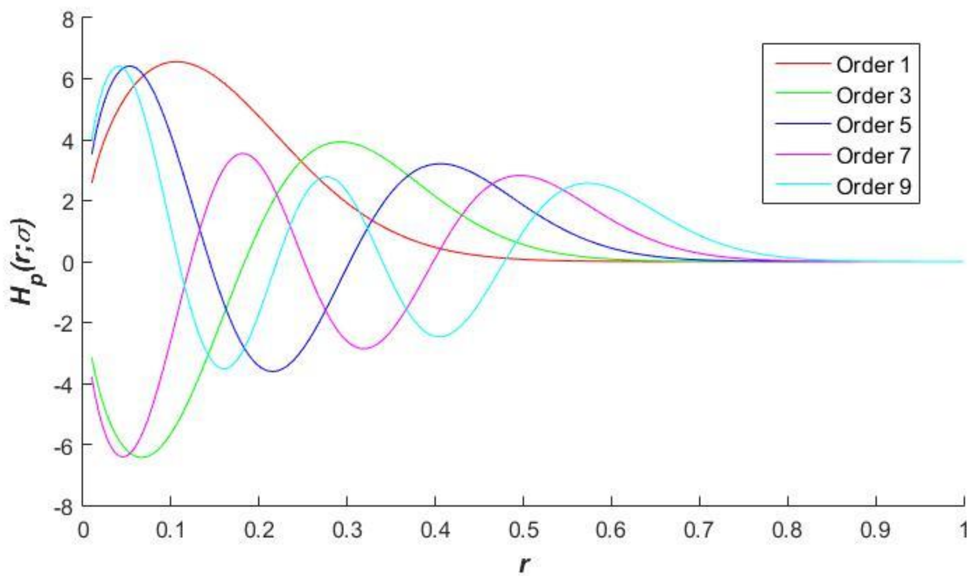
$$\int_0^\infty \tilde{H}_m(r;\sigma)\tilde{H}_n(r;\sigma)rdr = \delta_{mn}$$

$$(m,n)\in\{0,2,4,\cdots\} \text{ or } (m,n)\in\{1,3,5,\cdots\}$$

Special for polar coordinates.

Plotting of RHPs





$$\sigma = 0.15$$

$$r \in [0,1]$$

New OGMs

- HFMs defined over disc
  - Basis functions

$$V_{nl}(r,\theta;\sigma)=\tilde{H}_n(r;\sigma)\cdot\exp(il\theta)$$

  - Moments

$$h_{nl}=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}f(r,\theta)V_{nl}^{*}(r,\theta;\sigma)rdrd\theta$$

  - Image reconstruction

$$\hat{f}_{P_{\max}}(r,\theta)=\sum_{n=0}^{P_{\max}}\sum_{l=-N_{\max}}^{N_{\max}}h_{nl}V_{nl}(r,\theta;\sigma)$$

Rotation Invariants

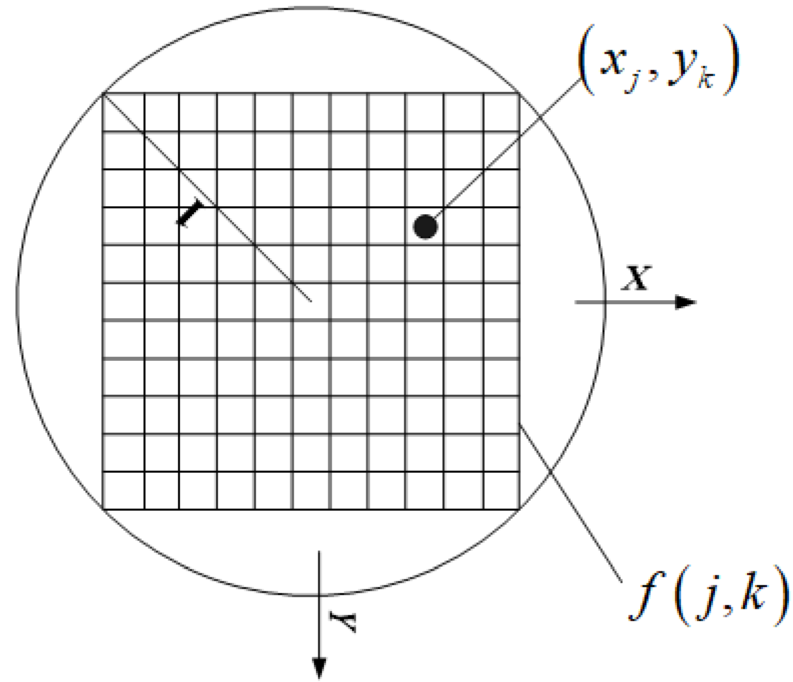
- Image $f(r,\theta)$ is rotated by α anticlockwise
  - Phase shift property

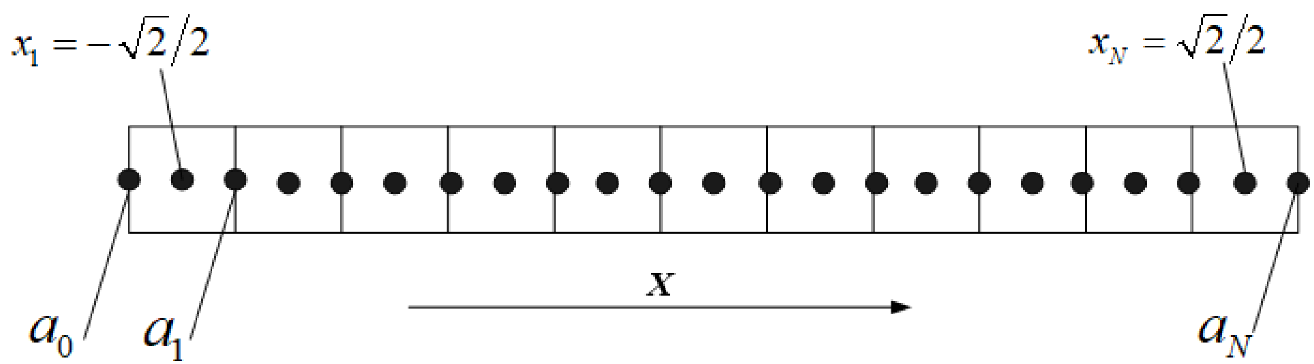
$$h_{nl}^{(\alpha)} = h_{nl} \cdot e^{-il\alpha}$$

  - Rotation invariants

$$v_{nl} = \frac{h_{nl}}{\left(h_{n_r l_r}\right)^{l/l_r}}, \quad \text{with } h_{n_r l_r} \neq 0$$

Discrete Implementation





Accurate Computation

- For digital image $f(j,k)$ with $N \times N$ pixels
  - Implementation in discrete case

$$h_{nl} = \frac{1}{2\pi} \sum_{j=1}^{N} \sum_{k=1}^{N} f(x_j, y_k) \Psi_{nl}(j,k), \; x_j^2 + y_k^2 \le 1$$

$$\text{where} \quad \Psi_{nl}(j,k) = \int_{a_j}^{a_{j+1}} \int_{b_k}^{b_{k+1}} V_{nl}^*(x, y; \sigma) dx dy$$

  - Gauss-Legendre numerical integration

$$\Psi_{nl}(j,k) = \frac{(a_{j+1} - a_j)(b_{k+1} - b_k)}{4} \sum_{u=1}^{M} \sum_{v=1}^{M} \omega_u \omega_v V_{nl}^*(x_u, y_v; \sigma)$$

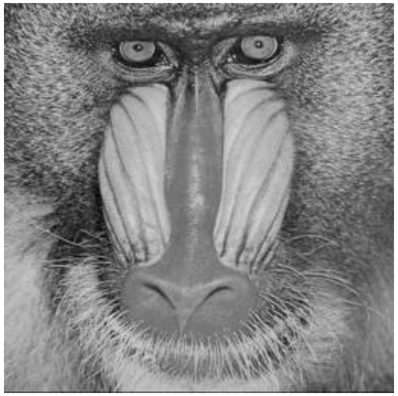
Constants

Image Reconstruction

- Methods: HFMs, Zernike (ZMs), Fourier-Mellin (FMMs), pseudo-Zernike (PZMs)
  - Criterion NMSE

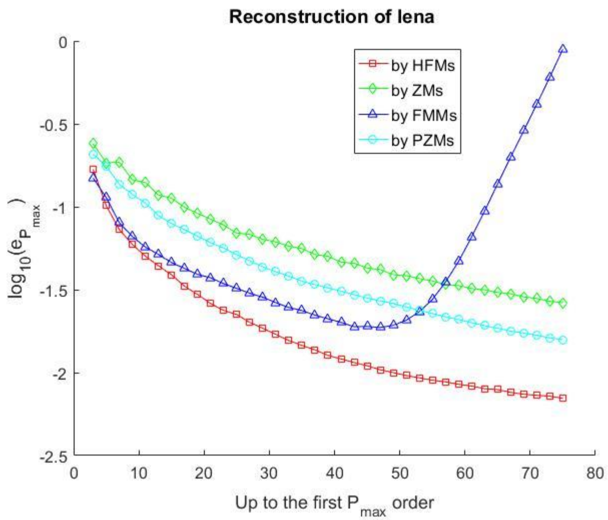
$$e_P = \frac{\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\hat{f}_P(j,k) - f(j,k)\right)^2}{\sum_{j=1}^{N}\sum_{k=1}^{N} f(j,k)^2}$$

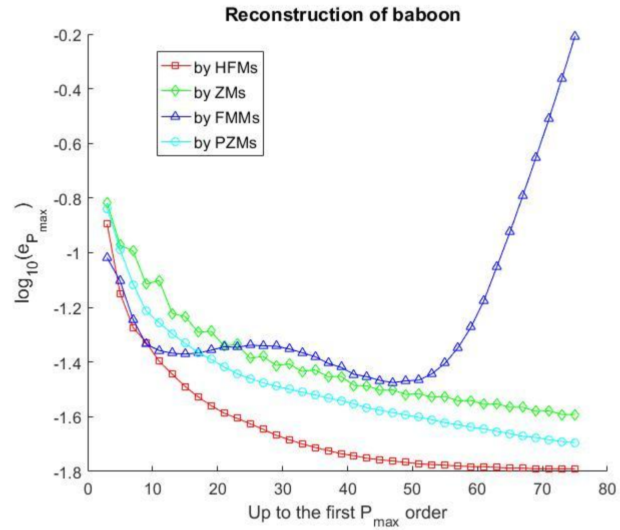
  - Tested benchmarks

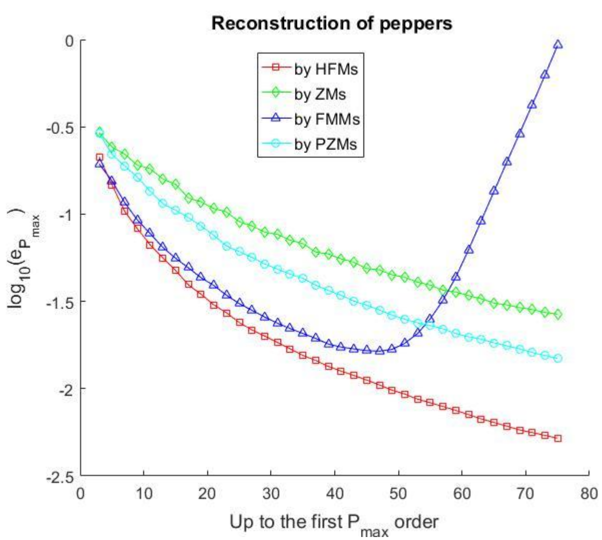



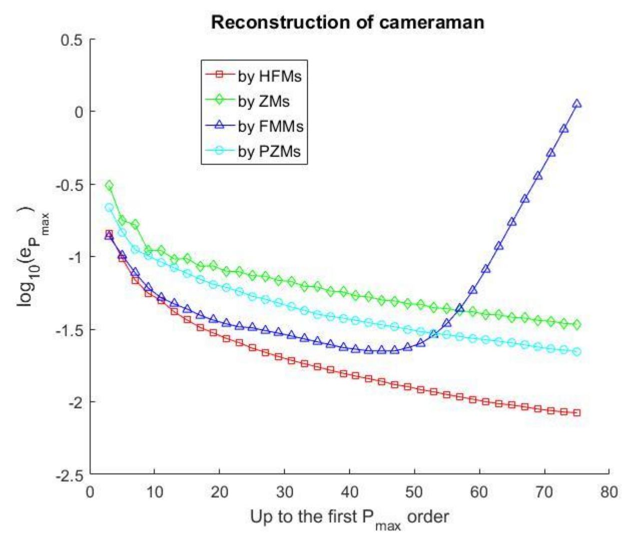
Experiments

Reconstruction of noise free images

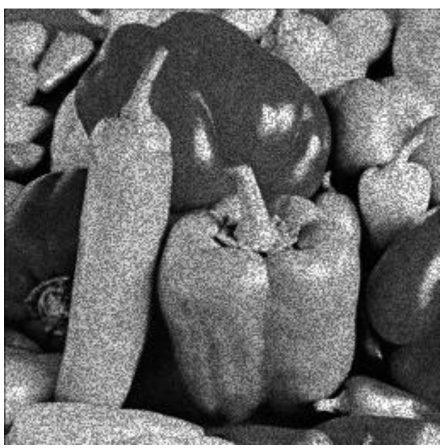


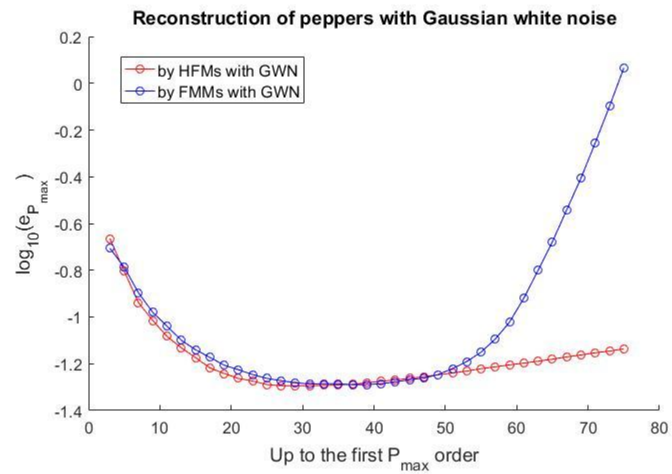


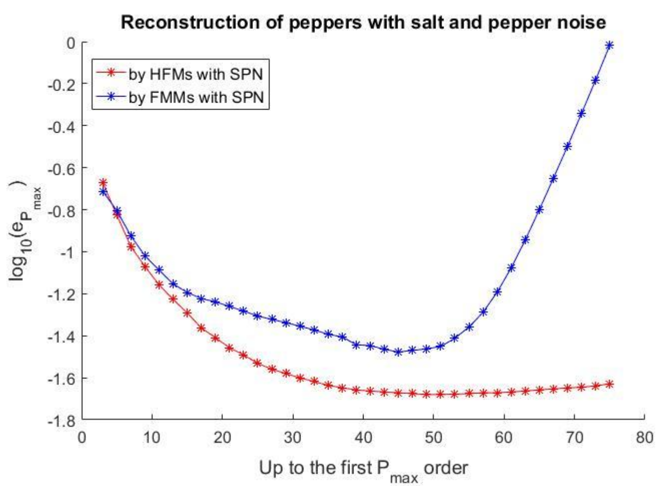


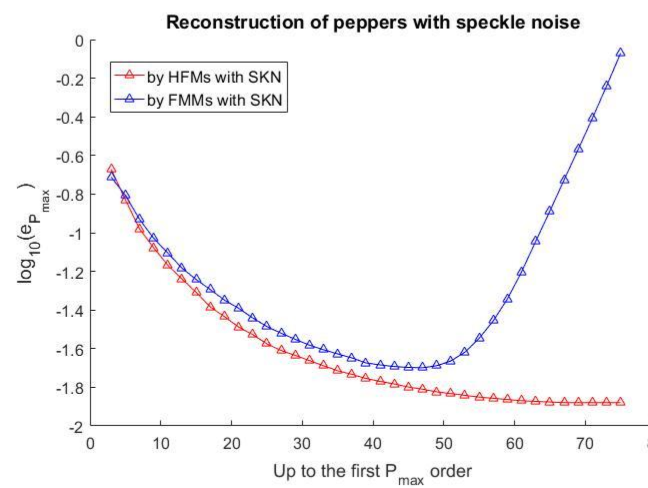


Reconstruction of noisy images

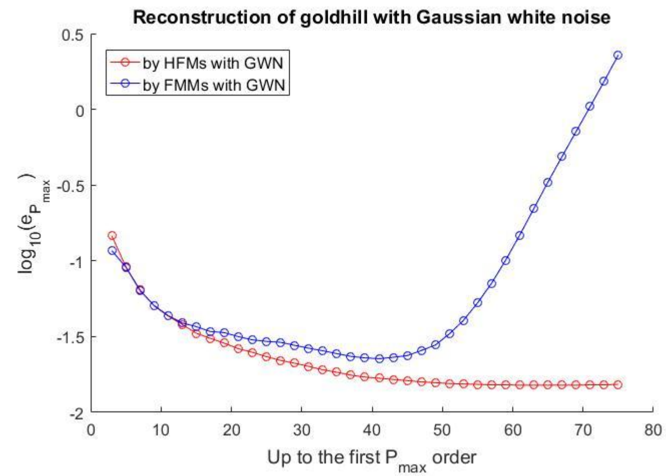


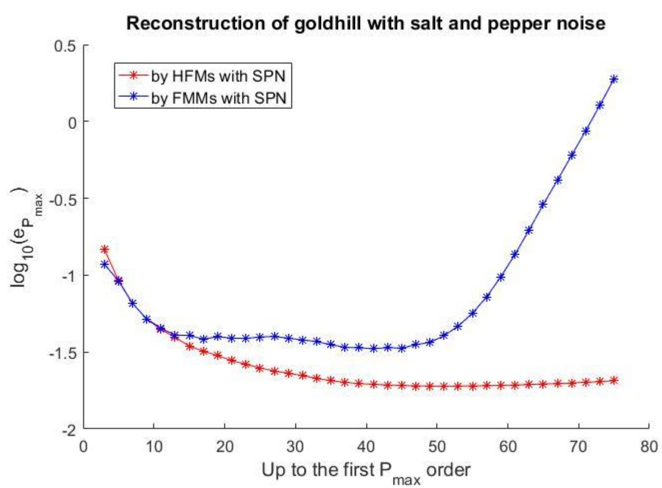


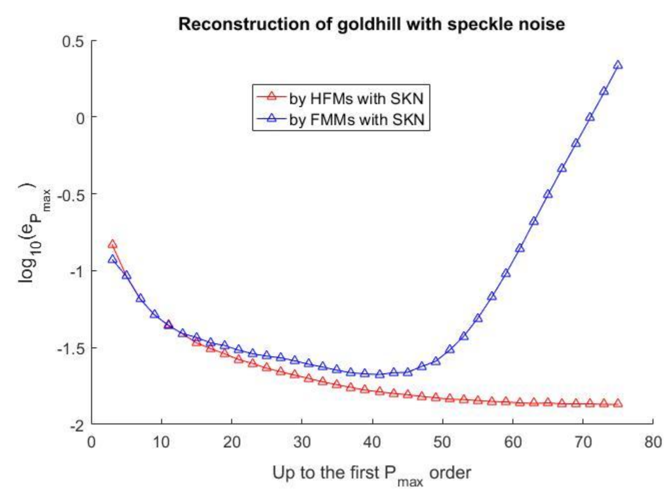




Reconstruction of noisy images







Template Matching

- Methods: Rotation invariants of HFMs and Geometric rotation invariants.
  - Reference image

Template Matching

- Feature
  - HFMs: the first 4 odd-order of HFMs (18 rotation invariants)
  - GEMs: Invariant bases of order 2 to 5 (18 rotation invariants)
- Matching result

Green rings: template locations
Red rings: templates found by HFMs
Blue rings: templates found by GEMs

Overlap means correct matching.

Summary

- Develop OGMs based on **Parity** of polynomials
- Define a new class of OGMs called **HFMs**
- Test image representation ability via image reconstruction
- Test discrimination power by invariant template matching

Thanks !