The discrete cosine transform on triangles

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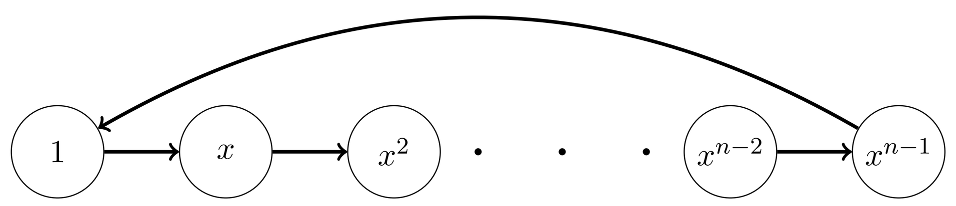
Algebraic signal processing



An algebraic signal model [1] consists of

- an algebra $\mathcal{A}$ of filters, capturing putting in series, in parallel, and amplificiation,
- an $\mathcal{A}$-module of signals, capturing filtering, adding of signals, and amplification,
- a bijection $\Phi\colon \mathbb{C}^n \longrightarrow M$ mapping data to the structure of the signal model.

For example in finite time discrete signal processing a set of numbers $s = (s_0, \dots, s_{n-1}) \in \mathbb{C}^n$ is called a signal if it is periodically extended, i.e. $s_N = s_{N \bmod n}$. The finite z-transform is

$$(s_0, \dots, s_{n-1}) \mapsto \sum_{i=0}^{n-1} s_i x^i,$$

i.e. a map

$$\Phi\colon \mathbb{C}^n \longrightarrow \mathbb{C}[x]/\langle x^n - 1\rangle.$$

The shift in the z-domain is realized as multiplication by x

$$x \cdot \Phi(s) = \sum_{i=0}^{n-1} s_{i-1 \bmod n} x^i.$$

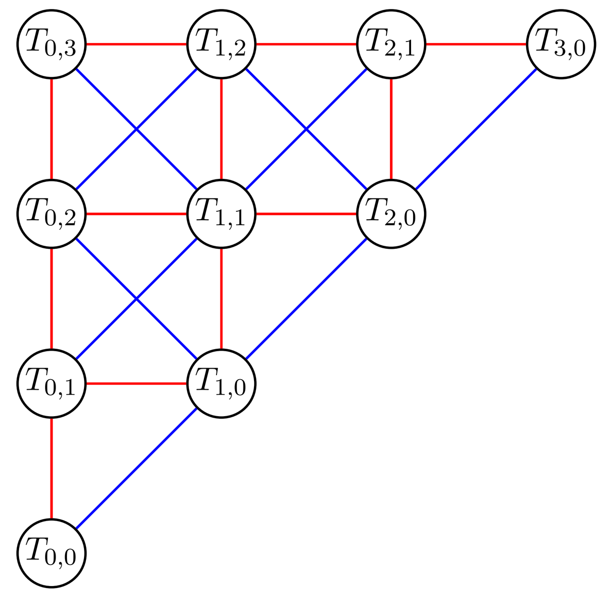
Filters are polynomials in the z-domain, as well. Filtering is just multiplication in $\mathbb{C}[x]/\langle x^n - 1\rangle$, i.e. circular convolution. So the filter algebra is $\mathcal{A} = \mathbb{C}[x]/\langle x^n - 1\rangle$ and the signal module is $M = \mathbb{C}[x]/\langle x^n - 1\rangle$. The polynomial $x^n - 1 = \prod_k x - e^{2\pi \mathrm{i} k/n}$ factors and any matrix realizing the isomorphism obtained by the Chinese remainder theorem

$$\mathbb{C}[x]/\langle x^n - 1\rangle \longrightarrow \bigoplus_l \mathbb{C}[x]/\langle x - e^{2\pi \mathrm{i} k/n}\rangle$$

is called a Fourier transform. Choosing bases one can for example obtain the discrete Fourier transform matrix

$$\mathsf{DFT}_n = \left[e^{2\pi \mathrm{i} kj/n}\right]_{k,j}.$$

Visualization graph



Orthogonal transform

By the multivariate Christoffel-Darboux formula

$$\mathcal{F}_n^\top \cdot H_n^\oplus \cdot \mathcal{F}_n = \mathsf{diag}\Big(\mathbb{T}_{n-1}^\top(\alpha) H_{n-1}^{-1} A_{n-1,1} \tfrac{\partial}{\partial x_1} \mathbb{T}_n(\alpha)\Big).$$

using the matrix $H_n^\oplus = \bigoplus_{k=0}^{n-1} H_k^{-1}$. Hence with

$$D_n = \mathsf{diag}\Big(\big(\mathbb{T}_{n-1}^\top(x) H_{n-1}^{-1} A_{n-1,1} \tfrac{\partial}{\partial x_1} \mathbb{T}_n(x)\big)^{-1}\Big)$$

one obtains an orthogonal transform matrix as

$$\mathcal{F}_n^{\mathsf{orth}} = \sqrt{H_n^\oplus} \mathcal{F}_n \sqrt{D_n}.$$

Algebraic signal model on triangles

The algebraic signal model on triangles is defined by $\mathcal{A} = \mathbb{C}[x_1, x_2]/\langle \mathbb{T}_n\rangle$, $M = \mathcal{A}$, and $\Phi\colon s \mapsto \sum_{k+\ell<n} s_{k,\ell} T_{k,\ell}$. The Fourier transform matrix of the signal model is

$$\mathcal{F}_n = (T_{k,\ell}(\alpha))_{k+\ell<n,\alpha},$$

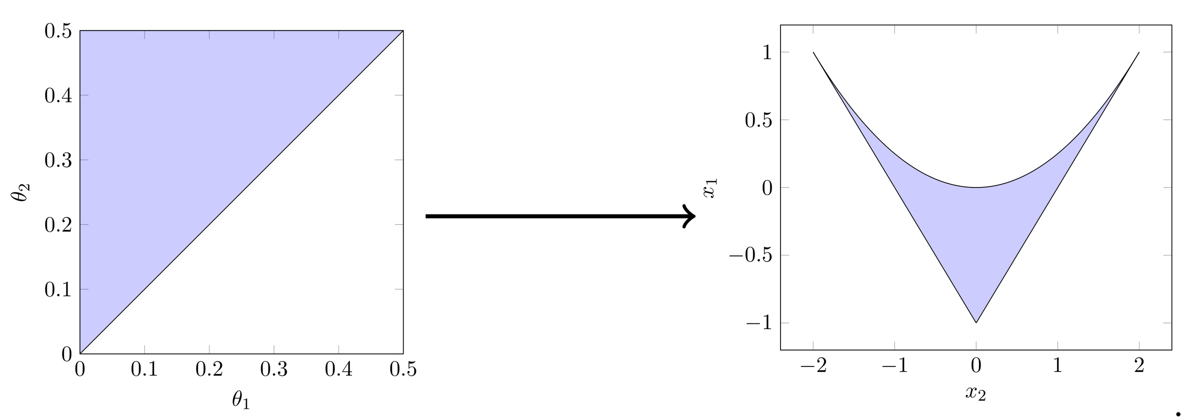
with α common zeros of the $\mathbb{T}_n$ and resembles the discrete cosine transform.

B_2-Chebyshev polynomials

Chebyshev polynomials of type B_2 obey the reccurence relation

$$x_1 \cdot T_{k,\ell} = \tfrac{1}{4}(T_{k+1,\ell} + T_{k-1,\ell} + T_{k-1,\ell+2} + T_{k+1,\ell-2}),$$
$$x_2 \cdot T_{k,\ell} = \tfrac{1}{4}(T_{k,\ell+1} + T_{k,\ell-1} + T_{k-1,\ell+1} + T_{k+1,\ell-1}).$$

They are orthogonal on the image of $F = \{\theta \in \mathbb{R}^2 \mid 0 \leq \theta_1 \leq \theta_2 \leq \frac{1}{2}\}$ under the map



The common zeros of $\mathbb{T}_n$ are given in θ-coordinates as

$$\{(\tfrac{k}{2n}, \tfrac{j}{4n}) \mid k = 0, \dots, n-1;$$
$$j = 1, 3, \dots, 2n-1; j \geq 2k\}.$$

Christoffel-Darboux formula

Let $\mathbb{T}_k = (T_{0,k}, T_{1,k-1}, \dots, T_{k,0})^\top$. Then there exist matrices $A_{k,i}$, $B_{k,i}$, and $C_{k,i}$ such that one has a recurrence relation

$$x_i \mathbb{T}_k = A_{k,i} \mathbb{T}_{k+1} + B_{k,i} \mathbb{T}_k + C_{k,i} \mathbb{T}_{k-1}.$$

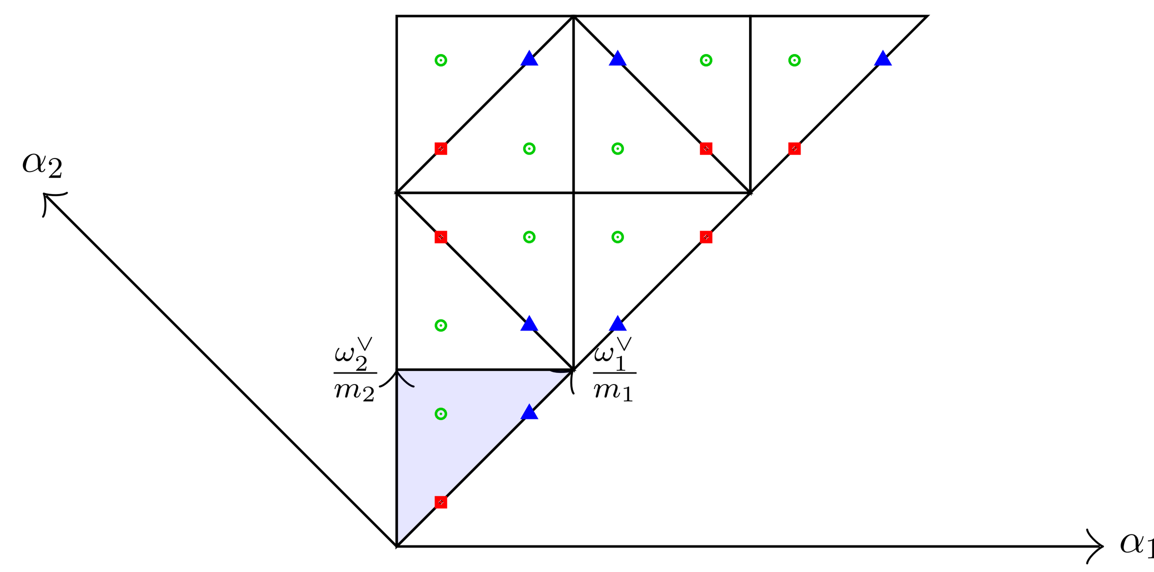
From the recurrence relation one can deduce a multivariate Christoffel-Darboux formula [2]

$$\sum_{k=0}^{n-1} \mathbb{T}_k^\top(x) H_k^{-1} \mathbb{T}_k(y) = \begin{cases} (x_i - y_i)^{-1} \cdot \\ \Big(\big(A_{n-1,i}\mathbb{T}_n(x)\big)^\top H_{n-1}^{-1} \mathbb{T}_{n-1}(y) \\ \quad - \mathbb{T}_{n-1}^\top(x) H_{n-1}^{-1} A_{n-1,i} \mathbb{T}_n(y)\Big) & \text{if } x_i \neq y_i \\ \mathbb{T}_{n-1}^\top(x) H_{n-1}^{-1} A_{n-1,i} \frac{\partial}{\partial x_i} \mathbb{T}_n(x) \\ \quad - \big(A_{n-1,i}\mathbb{T}_n(x)\big)^\top H_{n-1}^{-1} \frac{\partial}{\partial x_i} \mathbb{T}_{n-1}(x) & \text{if } x_i = y_i, \end{cases}$$

with matrices $H_0 = \frac{1}{2}$ and $H_k = \mathsf{diag}(\frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{16}, \frac{1}{8})$.

Fast algorithm

A fast algorithm for this transform exists and is based on a geometric stretching and folding operation. See [3] for details.



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