Sparse Recovery and Non-stationary Blind Demodulation

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Introduction

- **Sparse Recovery**

$$y = \mathrm{A}c$$

- Recover the unknown, sparse signal c from its low dimension measurements, y. The dictionary matrix A is a known, fat matrix. There could be more than one solution.

- **Blind Demodulation**

$$y = \mathrm{DA}c$$

- The measurements undergo an additional modulation process. Recover the unknown, diagonal modulation matrix, D, is referred to as blind demodulation.

- **Applications: Blind super-resolution, self-calibration, etc.**

Signal Model

- In this paper, we consider a general sparse recovery and blind demodulation model. Different from the ones in the literature, in our general model, each dictionary atom undergoes a distinct modulation process; we refer to this as non-stationary modulation.

- **Sparse Recovery and Non-stationary Blind Demodulation**

$$y = \sum_{j=1}^{M} c_j \mathrm{D}_j a_j + n \in C^N \qquad (1)$$

- a_j is the j-th column of the known dictionary matrix A.
- D_j is the modulation matrix of the j-th dictionary atom.
- c_j is the j-th entry of the unknown sparse vector c.
- n is the unknown additive noise.

- **Subspace Assumption**

$$\mathrm{D}_j = \mathrm{diag}(\mathrm{B}h_j) \in C^{N\times N} \qquad (2)$$

- B in $C^{N\times K}$ is the known subspace matrix.
- h_j is the unknown coefficient vector.

- **Lifting Technique**

$$\begin{cases} \mathrm{X} = [\, c_1 h_1 \;\; c_2 h_2 \;\; \cdots \;\; c_M h_M \,] \in C^{K\times M} \\ y = L(\mathrm{X}) \end{cases} \qquad (3)$$

Main Theorems

- When there is no noise, we solve the following equality constrained, block L1 norm optimization problem.

$$\underset{\mathrm{X}\in C^{K\times M}}{\text{minimize}} \|\mathrm{X}\|_{2,1} \quad \text{subject to } y = L(\mathrm{X}) \qquad (4)$$

Theorem I (Noiseless Case)

Consider the observation model in equation (1), assume that n=0, at most J(< M) coefficients c_j are nonzero, and furthermore assume that the nonzero coefficients c_j are real-valued and positive. Suppose that each modulation matrix D_j satisfies the subspace constraint (2), where $\mathrm{B}^H\mathrm{B} = I_K$ and each h_j has unit norm.

Then the solution X to problem (4) is the ground truth solution X_0—which means that c_j, h_j, and D_j can all be successfully recovered for each j —with probability at least 1-$O(N^{-\alpha+1})$, if A is a real, random Gaussian matrix and

$$\frac{N}{\log^2(N)} \geq C_\alpha \mu_{MAX}^2 KJ\left[\log(M-J) + \log(N)\right].$$

Here C_α is a constant defined for α>1 and the coherence parameter

$$\mu_{MAX} = \max_{i,j} \sqrt{N}\,|\mathrm{B}_{ij}|.$$

- When the measurements are contaminated with bounded noise, we solve the inequality constrained, block L1 norm optimization problem.

$$\underset{\mathrm{X}\in C^{K\times M}}{\text{minimize}} \|\mathrm{X}\|_{2,1} \quad \text{subject to} \|y - L(\mathrm{X})\|_2 \leq \eta \qquad (5)$$

Theorem II (Noisy Case)

If the measurements are contaminated with the bounded noise, $\|n\|_2 \leq \eta$, then with probability at least 1-$O(N^{-\alpha+1})$, the solution X to problem (5) satisfies

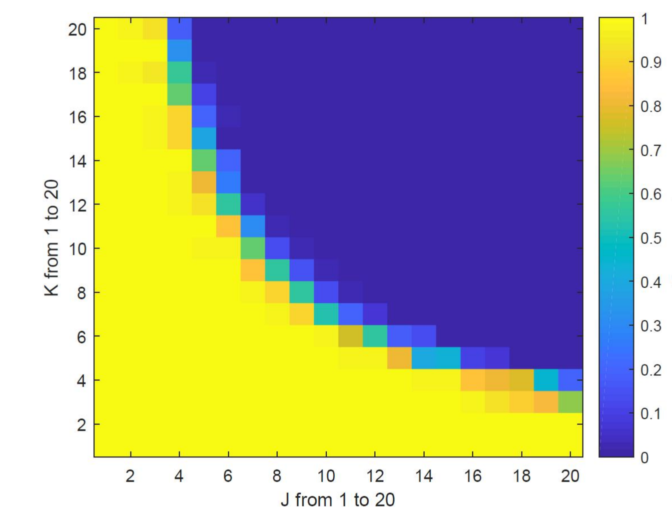
$$\|\mathrm{X}-\mathrm{X}_0\|_F \leq \left(C_1 + C_2\sqrt{J}\right)\eta$$

when

$$\frac{N}{\log^2(N)} \geq C_\alpha \mu_{MAX}^2 KJ\left[\log(C\mu_{MAX}\sqrt{KJ})C + 1\right] \cdot \left[\log(M-J) + \log(MK) + \log(N)\right]$$

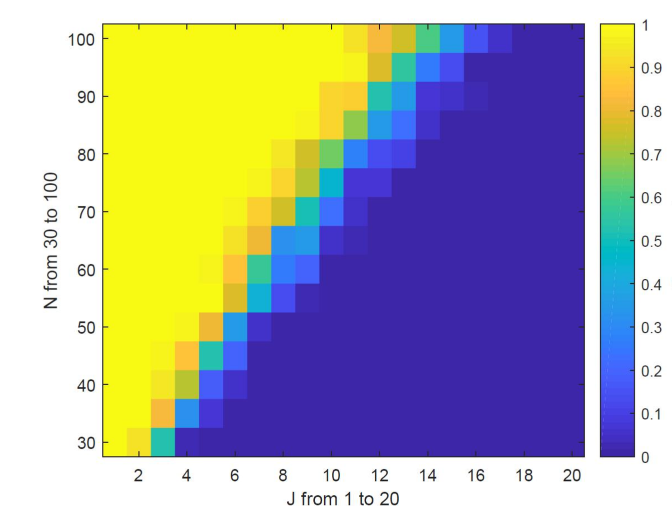
where C, C_1, and C_2 are constant. C_α is a constant defined for α>1.

Numerical Simulations

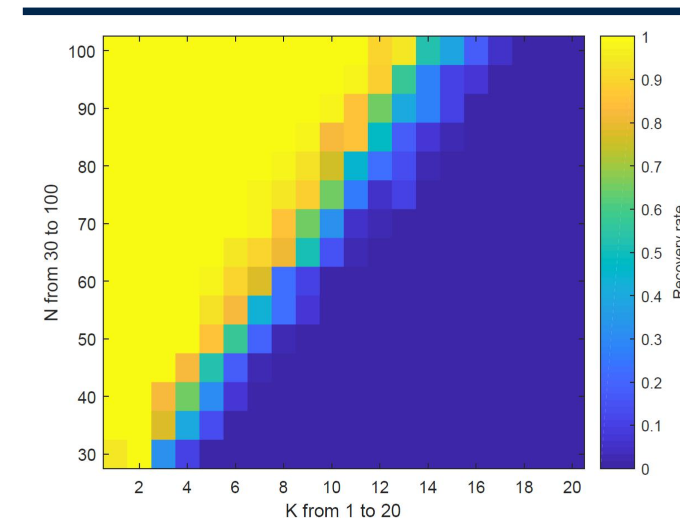


The relation between the subspace dimension of the modulation matrix, K, and the number of committed atoms, J, in terms of the success recovery rate.

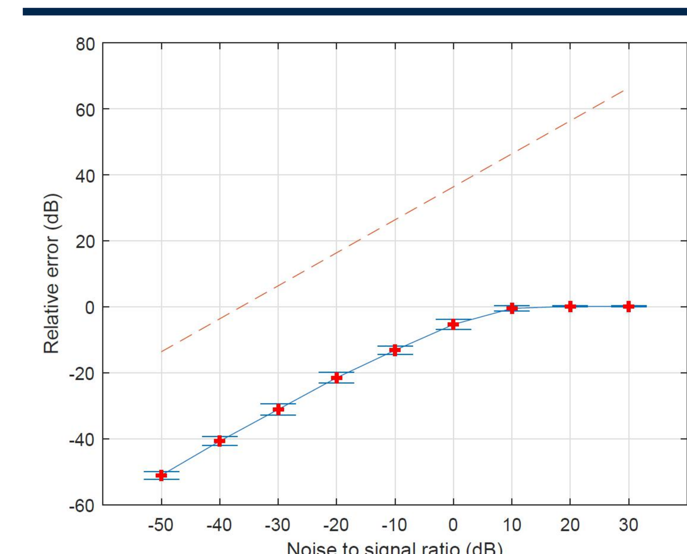
Success recovery: $\frac{\|\mathrm{X}-\mathrm{X}_0\|_F}{\|\mathrm{X}_0\|_F} \leq 10^{-5}$



The nearly linear relation between the dimension of the observed signal, N, and the number of committed atoms, J, in terms of the success recovery rate.



The nearly linear relation between the dimension of the observed signal, N, and the subspace dimension, K, in terms of the success recovery rate.



The relation between the relative error (dB) and noise to signal ratio (dB). The blue horizontal sticks and red plus sign indicate the range of the standard deviation and the meanof the relative error (dB) given a specific noise to signal ratio (dB). The dashed line shows the theoretical error bound from Theorem II.

$$20\log_{10}\left(\frac{\|\mathrm{X}-\mathrm{X}_0\|_F}{\|\mathrm{X}_0\|_F}\right) \leq 20\log_{10}\left(\frac{\|n\|_2}{\|\mathrm{X}_0\|_F}\right) + 20\log_{10}(C)$$

Conclusion

- Introduce the general sparse recovery and non-stationary blind demodulation signal model.
- Propose to solve the model via the constrained block L1 norm optimization problem.
- Derive the near optimal, sufficient sample complexity for success recovery in the noiseless case and bound the recovery error in the noisy case.