One-Bit Unlimited Sampling

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Introduction & Contribution

Key Takeaways

- Unlike Shannon's Sampling Theorem, the analog-to-digital converters (ADCs) are limited in dynamic range, thus prone to saturation and clipping. In order to circumvent these problems, the authors introduced the concept of **Unlimited Sampling** in [1].
- Behind work [1] are recent developments in ADC design - the **Self-Reset ADCs**, which compute modulo samples [2].
- The **Unlimited Sampling Theorem** proves that a bandlimited signal can be perfectly recovered from modulo samples. The sampling rate is independent of the ADC threshold.
- As a step towards practical implementation, we consider not only sampling, but also quantization.
- We combine the advantages of Unlimited Sampling and **One-Bit Sigma-Delta Quantization** (SDQ) to obtain an ADC scheme that has low complexity due to coarseness of quantization and at the same time overcomes the dynamic range limitations of conventional One-Bit SDQ.

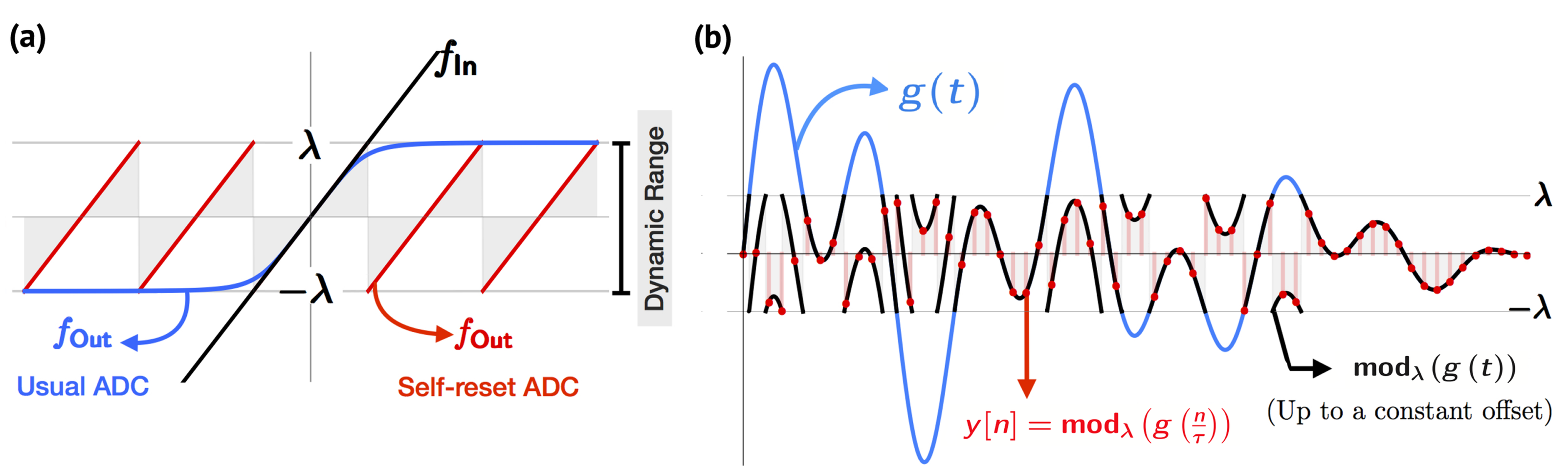
Unlimited Sampling of Bandlimited Functions

- Let $\tau \geqslant 1$ be the (over)sampling rate and $g(t)$ be a π-bandlimited function.
- In Unlimited Sampling framework, we sample g using non-linear principle:

$$y[n] = \mathrm{mod}_\lambda\left(g\left(\tfrac{n}{\tau}\right)\right), \quad n \in \mathbb{Z}, \quad \tau \geqslant 1$$

- Such folded samples are acquired using a version of the Self-Reset ADC [2].
- Even if $g(t) \gg \lambda$, $y[n] \in [0, \lambda)$. In this work, we set $\lambda = 1$.

Unlimited Sampling in Action



(a) Usual ADC compared with self-reset ADC. In usual ADC, whenever the input signal f_{In} voltage exceeds some λ, the output signal f_{Out} saturates and this resuts in clipping. In contrast, the self-reset ADC folds f_{In} such that f_{Out} is always in the range $[-\lambda, \lambda]$. **(b)** For π-bandlimited function g we plot the continuous version of self-reset ADC, $\mathrm{mod}_\lambda(g(t))$, together with uniform samples $y[n]$.

Unlimited Sampling Theorem [1]

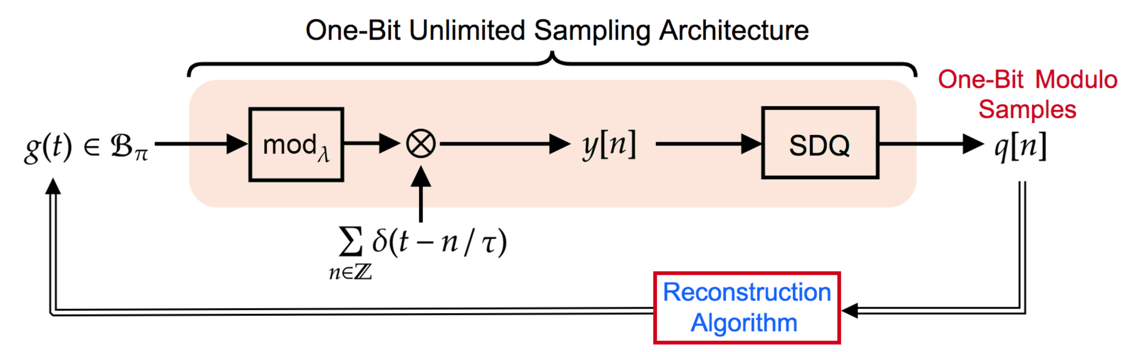
A sufficient condition for recovery of π-bandlimited signal g from its modulo samples $y[n] = \mathrm{mod}_\lambda\left(g\left(\frac{n}{\tau}\right)\right)$ up to additive multiples of 2λ is $\tau > \pi e$.

Unlimited Sampling Meets One-Bit Quantization

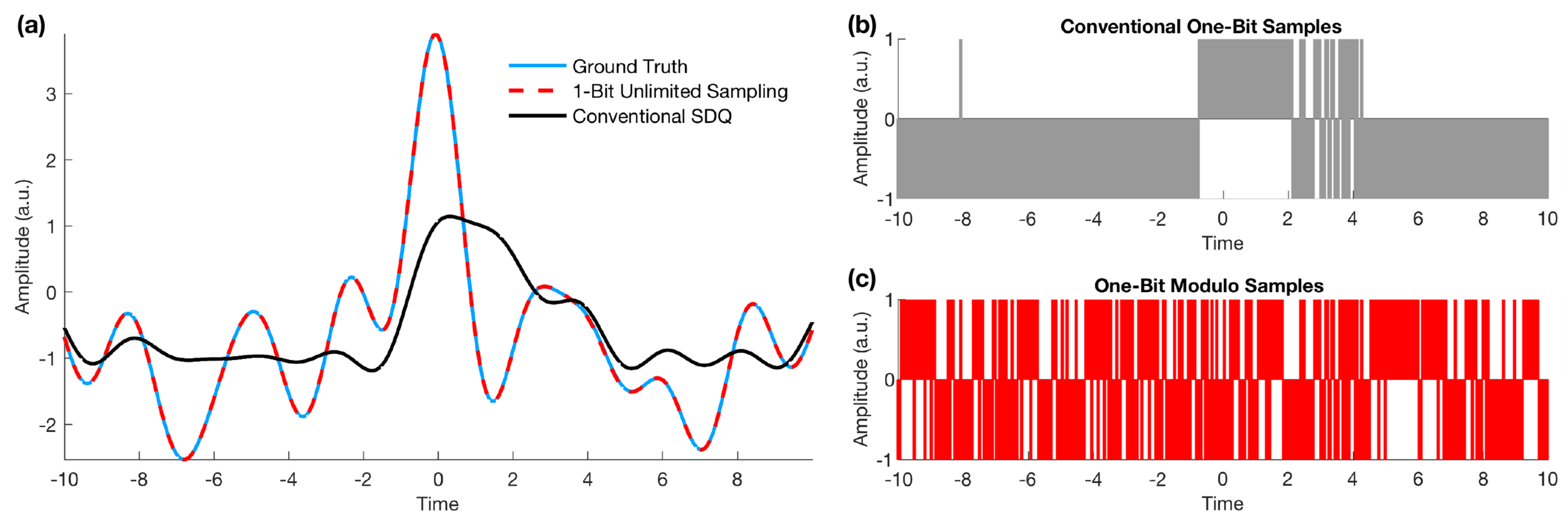
- In order to discretize the range, $y[n]$ is quantized via the first order, one-bit **Sigma-Delta Quantizer**:

$$u[n] = u[n-1] + y[n] - q[n], \quad q[n] = \mathrm{sign}\left(u[n-1] + y[n]\right)$$

- System architecture for One-Bit Unlimited Sampling:



Conventional One-Bit Sampling vs One-Bit Unlimited Sampling



(a) Conventional one-bit sampling leads to reconstruction failure while our method allows for fair reconstruction. **(b)** Conventional one-bit samples exhibit saturation if the dynamic range exceeds [-1,1]. **(c)** Due to amplitude folding, one-bit modulo samples capture sufficiently more information about the signal than conventional one-bit samples.

Recovery from One-Bit Modulo Samples

Modular Decomposition [1]

Function $g \in \mathcal{B}_\pi$ admits a decomposition $g[n] = y[n] + \varepsilon_g[n]$, $\varepsilon_g[n] \in 2\lambda\mathbb{Z}$.

- With SDQ involved, we decompose not g, but its multi-bit representation: $q_{\mathrm{MB}}[n] = q[n] + \varepsilon_g[n]$. Recovering $q_{\mathrm{MB}}[n]$ boils down to finding $\varepsilon_g[n]$.
- Consider smoothing kernel $\psi^N(t) := \mathrm{B}^N\left(\frac{N}{2}t\right)/\max\left(\mathrm{B}^N(t)\right)$, where B^N is a B-spline of order N, and its sampled version $\psi_h^N[n]$ with sampling rate $h \in 2\mathbb{N}$.

Recovery Algorithm

Input: $q[n]$, $\psi_h^N[n]$ and $\beta_g \geq \|g\|_{L^\infty}$.
Output: $\tilde{g}(t) \approx g(t)$.

1. Compute $(\Delta q * \psi_h^N)[n]$.
2. Compute $\mathrm{mod}_1\left((\Delta q * \psi_h^N)[n]\right) - (\Delta q * \psi_h^N)[n]$ and retain one point from each of its non-zero neighborhoods to obtain $\Delta\tilde{\varepsilon}_g[n]$.
3. Apply summation operator to obtain $\tilde{\varepsilon}_g[n]$.
4. Compute $\tilde{q}_{\mathrm{MB}}[n] = q[n] + \tilde{\varepsilon}_g[n]$.
5. Reconstruct $\tilde{g}(t)$ from $\tilde{q}_{\mathrm{MB}}[n]$ via low-pass filter.

Sufficiency Condition and Error Bound

One-Bit Unlimited Sampling Theorem

Given

- $g \in \mathcal{B}_\pi$ and not superoscillating, $\beta_g \geqslant \|g\|_{L^\infty}$,
- $q[n]$ - the one-bit modulo samples of $g(t)$,
- $\psi_h^N[n]$ - the samples of the smoothing kernel $\psi_h^N(t)$ with sampling rate $h_r := 2\left\lceil \|\partial^2\psi_h^N\|_{L^1}\right\rceil$,
- a valid reconstruction kernel $\varphi(t)$,

a sufficient condition for approximate recovery of $\tilde{g}(t)$ from $q[n]$ (up to additive multiples of 2) is

$$\tau > 4\pi e \beta_g \left(\left\lceil \|\partial^2\psi_h^N\|_{L^1}\right\rceil \|\psi_h^N\|_{L^1} + 1\right).$$

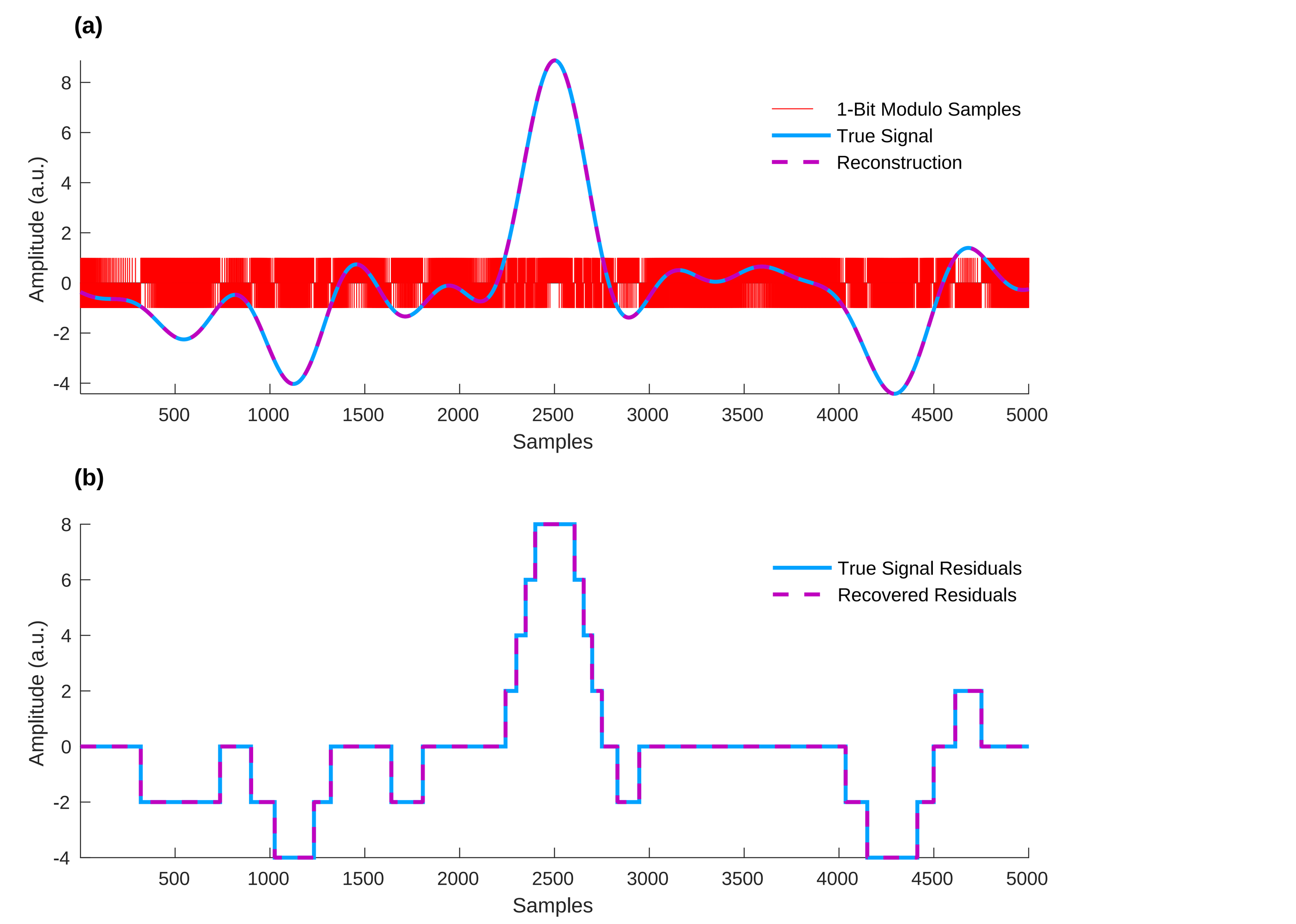
Under these conditions, Recovery Algorithm yields the reconstruction error

$$|g(t) - \tilde{g}(t)| \leqslant \frac{1}{\tau}\left(\|\partial\varphi\|_{L^1} + M(\varphi, \psi_h^N)\right),$$

where $M(\varphi, \psi_h^N)$ is a constant dependent on the choice of kernels φ and ψ_h^N.

- Our algorithm allows for recovery with accuracy $O(1/\tau)$, which is close to the best known error bound $O(\tau^{-3/2})$ for conventional first order SDQ [3].

Reconstruction Example



(a) Randomly generated π-bandlimited signal g, its one-bit modulo samples $q[n]$ acquired with $\tau = 250$ and the reconstructed signal $\tilde{g}$ which is obtained using second order ψ^2. The mean error $|g - \tilde{g}|$ is 2.1×10^{-3}. **(b)** The true residual $\varepsilon_g[n]$ and its approximate recovery $\tilde{\varepsilon}_g[n]$.

References

[1] A. Bhandari, F. Krahmer, and R. Raskar. On unlimited sampling. In proc. *SampTA 2017*, 2017.
[2] J. Rhee and Y. Joo. Wide dynamic range CMOS image sensor with pixel level ADC. *Electron. Lett.*, 39(4):360-361, 2003.
[3] C. Güntürk. Approximating a bandlimited function using very coarsely quantized data: improved error estimates in Sigma-Delta modulation. *J. Amer. Math. Soc.*, 17(1):229-242, 2004.