On the Computability of the Secret Key Capacity Under Rate Constraints

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joint work with
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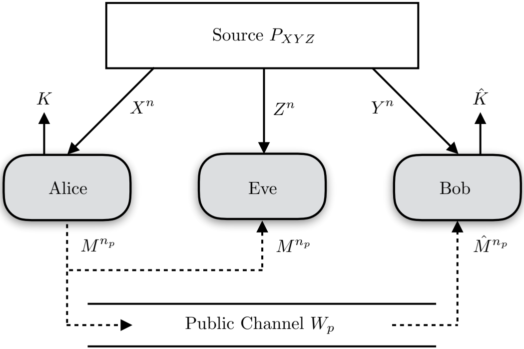
Secret Key Generation



- Based on X^n and Y^n, Alice and Bob want to generate a secret key K
- Single forward transmission of helper data M^{n_p} over the public channel
  - Noisy W_p imposes a rate constraint $R_\mathsf{p} = C(W_\mathsf{p})$
  - Noiseless W_p results in no rate constraint
- Eve intercepts M^{n_p} error-free (worst case from a security perspective)

Forward SK Capacity

Theorem: [Csiszár/Narayan '00], [Bassi *et al.* '16]

The *forward SK capacity* $C_{\mathsf{SK}}(W_{\mathsf{p}}, P_{XYZ})$ for source P_{XYZ} and noisy public channel W_{p} is

$$\boldsymbol{C_{\mathsf{SK}}(W_{\mathsf{p}}, P_{XYZ}) = \max_{U,V} \left[I(V;Y|U) - I(V;Z|U) \right]}$$

where U and V are auxiliary random variables that satisfy the Markov chain relation $U - V - X - (Y, Z)$ and further satisfy the rate constraint

$$I(V;X|Y) \leq C(W_{\mathsf{p}}).$$

Moreover, it may be assumed that $V = (U, V')$ where the cardinalities of the alphabets of both U and V' are at most $|\mathcal{X}| + 1$.

I. Csiszár and P. Narayan, "Common randomness and secret key generation with a helper," *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 344–366, Mar. 2000

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Forward SK Capacity (2)

- W_{p} becomes noiseless for $R_{\mathsf{p}} = C(W_{\mathsf{p}}) \to \infty$ and R_{p} is inactive

Corollary: [Ahlswede/Csiszár '93]

The *forward SK capacity* $C_{\mathsf{SK}}(P_{XYZ})$ for source P_{XYZ} is

$$\boldsymbol{C_{\mathsf{SK}}(P_{XYZ}) = \max_{U,V} \left[I(V;Y|U) - I(V;Z|U) \right]}$$

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R. Ahlswede and I. Csiszár, "Common Randomness in Information Theory and Cryptography-Part I: Secret Sharing," *IEEE Trans. Inf. Theory*, vol. 39, no. 4, pp. 1121–1132, Jul. 1993

Gold Standard / Holy Grail

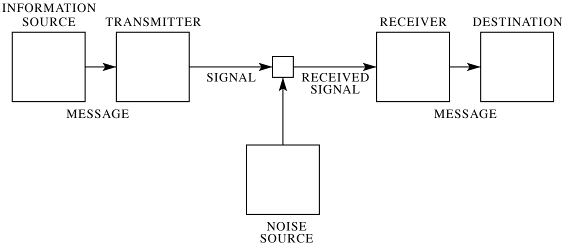
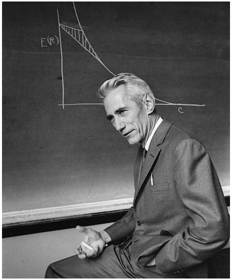


Fig. 1 — Schematic diagram of a general communication system.



The *capacity* $C(W)$ of a discrete memoryless channel (DMC) W is

$$\boldsymbol{C(W) = \max_{X} I(X;Y) = \max_{P_X} I(P_X, W)}$$

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Capacity of DMCs

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- *Entropic quantities*
- *Single-letter*
- *Convex optimization problem*

➠ Of particular relevance as it allows to compute the capacity $C(W)$ as a function of the channel W given by a convex optimization problem

What do we actually mean with "compute"?

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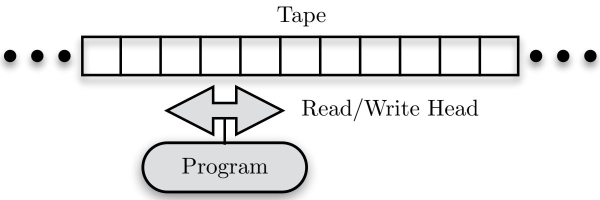
Turing Machine



Mathematical model of an abstract machine that manipulates symbols on a strip of tape according to certain given rules

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

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Turing Machine (2)

- Turing machines can simulate any given algorithm and therewith provide a simple but very powerful model of computation
- **No** limitations on computational complexity, unlimited computing capacity and storage, and execute programs completely error-free
- Extends to programming languages which are then called *Turing-complete*

➠ **Fundamental performance limits for today's digital computers**

➠ *Ideal concept to decide if the capacity can computed* **algorithmically** *(without putting any constraints on the computational complexity)*

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Computability

- Computable numbers are real numbers that are computable by Turing machines

- A sequence $\{r_n\}_{n\in\mathbb{N}}$ is called a *computable sequence* if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}$$

- Then a real number x is said to be *computable* if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ such that

$$|x - r_n| < 2^{-n}$$

➡ $\mathbb{R}_c$ is the set of computable real numbers

R. I. Soare, *Recursively Enumerable Sets and Degrees*. Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg, 1987

Computability (2)

➠ Based on this, we can define *computable probability distributions* and *computable channels*

- We define the set of computable probability distributions $\mathcal{P}_c(\mathcal{X})$ as the set of all probability distributions

$$P \in \mathcal{P}(\mathcal{X}) \text{ such that } P(x) \in \mathbb{R}_c,\ x \in \mathcal{X}$$

- Let $\mathcal{CH}_c$ be the set of all computable channels, i.e., for a channel

$$W : \mathcal{X} \to \mathcal{P}(\mathcal{Y}) \text{ we have } W(\cdot|x) \in \mathcal{P}_c(\mathcal{Y}) \text{ for every } x \in \mathcal{X}$$

Computability (3)

Definition: Borel Computability

A function $f : \mathbb{R}_c \to \mathbb{R}_c$ is called *Borel computable* if there is an algorithm that transforms each given computable sequence of a computable real x into a corresponding representation for $f(x)$.

- Turing's notion of computability conforms to Borel computability

Capacity $C(W) = \max_X I(X;Y)$ is Borel computable

Proof outline:

1. $x \log_2 x$, $x \in [0,1]$, is a Borel computable function
2. function $\sum x \log_2 x$ is computable
3. function $I(X;Y)$ is computable
4. $C(W) = \max_X I(X;Y)$ is a computable function since it is the maximum of computable functions ✓

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Computability (4)

- There are weaker forms of computability including *Markov computability* and *Banach-Mazur computability*

Definition: Markov Computability

A function $f : \mathbb{R}_c \to \mathbb{R}_c$ is called *Markov computable* if there is an algorithm that converts an algorithm for a computable real x into an algorithm for $f(x)$.

Definition: Banach-Mazur Computability

A function $f : \mathbb{R}_c \to \mathbb{R}_c$ is called *Banach-Mazur computable* if f maps any given computable sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers into a computable sequence $\{f(x_n)\}_{n=1}^{\infty}$ of real numbers.

J. Avigad and V. Brattka, "Computability and analysis: The legacy of Alan Turing," in *Turing's Legacy: Developments from Turing's Ideas in Logic*, R. Downey, Ed. Cambridge, UK: Cambridge University Press, 2014

Computability (4)

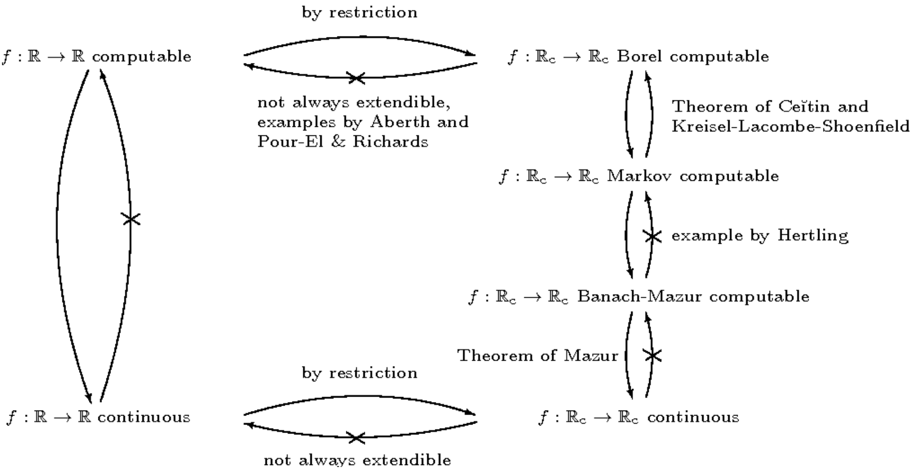

Figure 5 from

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Back to the Forward SK Capacity

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QUESTION: Can we compute these capacity expressions?

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QUESTION: Can we compute these capacity expressions?

Without Rate Constraint

Theorem:

For all $|\mathcal{X}| \geq 2$, $|\mathcal{Y}| \geq 2$, and $|\mathcal{Z}| \geq 2$, the forward SK capacity

$$C_{\mathsf{SK}}(P_{XYZ}) = \max_{U,V} \left[I(V;Y|U) - I(V;Z|U) \right]$$

without public rate constraint for source P_{XYZ} is Borel computable.

Proof outline:

1. $x \log_2 x$, $x \in [0,1]$, is a Borel computable function
2. function $\sum x \log_2 x$ is computable
3. functions $I(V;Y|U)$ and $I(V;Z|U)$ are computable
4. $C_{\mathsf{SK}}(P_{XYZ}) = \max_{U,V}[I(V;Y|U) - I(V;Z|U)]$ is a computable function since it is the maximum of computable functions ✓

- This can even be strengthened: Forward SK capacity $C_{\mathsf{SK}}(P_{XYZ})$ is a *computable continuous function*!

H. Boche, R. F. Schaefer, S. Baur, and H. V. Poor, "On the algorithmic computability of the secret key and authentication capacity under channel, storage, and privacy leakage constraints," 2018, submitted

With Rate Constraint

Theorem:

For all $|\mathcal{X}| \geq 2$, $|\mathcal{Y}| \geq 2$, and $|\mathcal{Z}| \geq 2$, the forward SK capacity $C_{\mathsf{SK}}(W_{\mathsf{p}}, P_{XYZ})$ for the source P_{XYZ} and the noisy public channel W_{p} is **not** Banach-Mazur computable.

Key ingredient:

- Proof by contradiction
- If $C_{\mathsf{SK}}(W_{\mathsf{p}}, P_{XYZ})$ would be Banach-Mazur computable, then the halting problem would be solvable!

➠ The forward SK capacity with rate-limited public public communication is not Banach-Mazur and therewith also **not** Turing computable!

ANSWER: The forward SK capacity is

- computable without public rate constraints and
- **non**-computable with public rate constraints!

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For all $|\mathcal{X}| \geq 2$, $|\mathcal{Y}| \geq 2$, and $|\mathcal{Z}| \geq 2$, the forward SK capacity $C_{\mathsf{SK}}(W_{\mathsf{p}}, P_{XYZ})$ for the source P_{XYZ} and the noisy public channel W_{p} is **not** Banach-Mazur computable.

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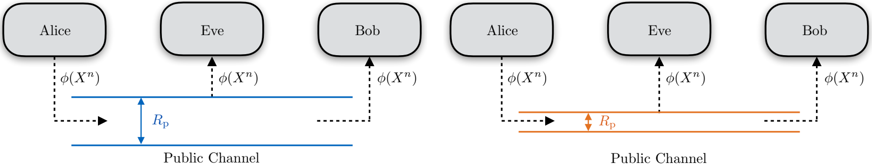
Conclusions

- Computability of the forward SK capacity has been studied
- Sharp phase transition between being computable and non-computable



- Rate constraint R_{p} inactive
- $C_{\mathsf{SK}}(P_{XYZ})$ computable (even computable continuous function)

- Rate constraint R_{p} active
- $C_{\mathsf{SK}}(R_{\mathsf{p}}, P_{XYZ})$ non-computable (not even Banach-Mazur)

Rate constraint on the public communication not only affects the performance, it is also turns an algorithmically non-tractable problem into a solvable problem!

Conclusions (2)

Thank you for your attention!

- Many extensions and other open problems including secure communication with active jammers [BSP '18], identification [BSP '18], secure authentication [BSBP '19], detection of denial-of-service attacks [BSP '19] and many others

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If time permits, another motivation for studying the computability....

Motivation

Status Quo[1]

- several 100.000 new software products every year
- all main products of top leading software companies are insecure, erroneous, and impossible to verify
- uncountable software updates to repair identified errors
- no secure infrastructure
- uncountable adversarial attacks on hardware and software (e.g. Vodafone monitors every 2ms an attack on their system)

[1] A. Schönbohm, President of Federal Office for Information Security (BSI), at *Technology Innovation 2018*, Feb 22, 2018

Verification

Industry with hardware and software products

⇒

Certification agency (e.g. BSI)

YES ⇒

Verified hard- and software infrastructure

⇓ **NO**

- **QUESTION:** How can we design this?

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Verification (2)

Implementation of solution of communication task:

Protocol
⊕ Physical channels
⊕ Attack classes

⇒

(universal) Turing machine to verify:

- security
- privacy
- efficiency
- ...

YES ⇒

Verified hard- and software infrastructure

⇓ **NO**

➠ Verification framework based on *Turing machines* and *computability* as developed in the beginning!

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Turing Machine (2)

Definition: Turing Machine

A Turing machine $\mathfrak{T}$ given by

$$\mathfrak{T} : \mathcal{CS} \times \mathcal{CH} \times \mathcal{CP} \times \mathbb{N} \to \{\text{yes / no}\}$$

is a mapping with

$$\mathfrak{T}(\mathrm{CS}, \mathrm{CH}, \mathrm{CP}, k) = \text{yes}$$

if and only if the performance requirements are satisfied and

$$C - R < \frac{1}{k} \tag{1}$$

where C denotes the capacity.

For a given communication scenario CS, the verification of security and spectral efficiency is called *effective*, if there exists a (universal) Turing machine such that for all valid channel inputs, communication protocols CP, and all $k \in \mathbb{N}$ the Turing machine always outputs the correct answer

Turing Machine (3)

- Within this framework, the task of the Turing machine is to answer the following two questions:

Question 1: Are all information theoretic performance requirements (such as probability of decoding error at the legitimate receiver or secrecy at an eavesdropper) satisfied?

Question 2: Is effectiveness given in the sense that the gap $1/k$ to the optimal performance limit can be controlled?

- In particular Question 2 requires that the **capacity C must be algorithmically computable**

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References II

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