When can a System of Subnetworks be Registered Uniquely?

Aditya V. Singh   Kunal N. Chaudhury

Department of Electrical Engineering
Indian Institute of Science

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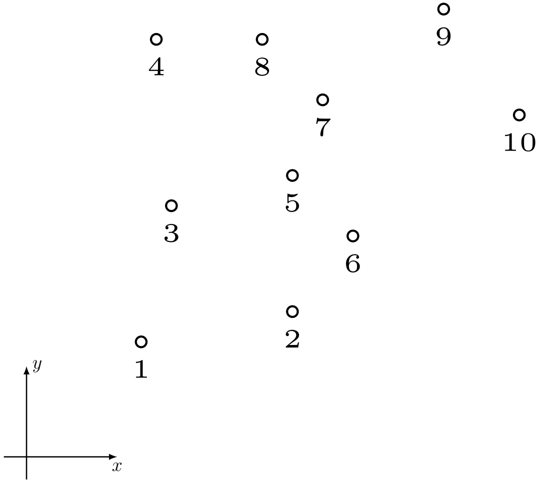
Sensor network localization



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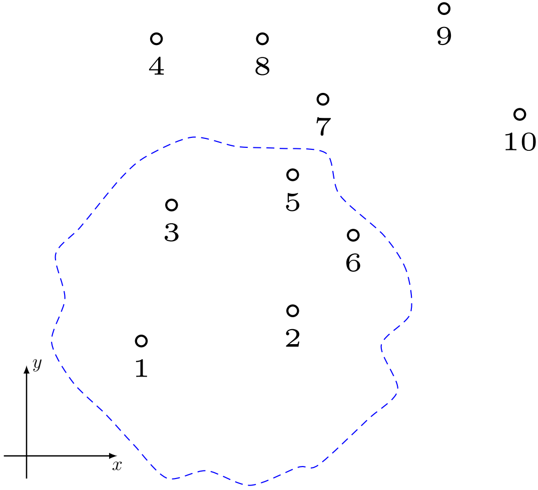

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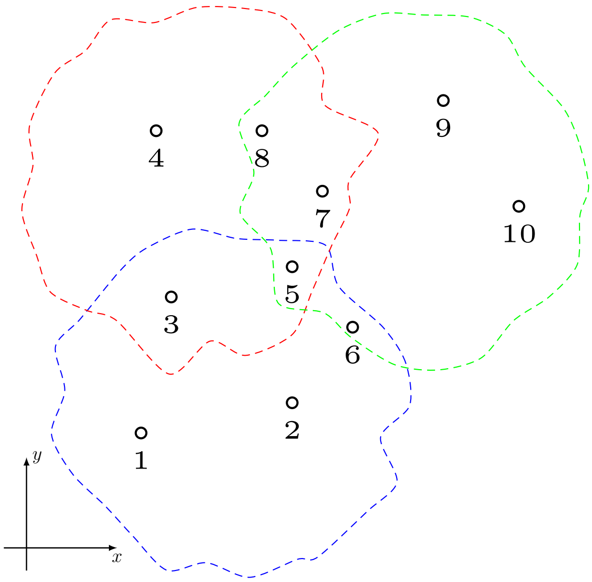

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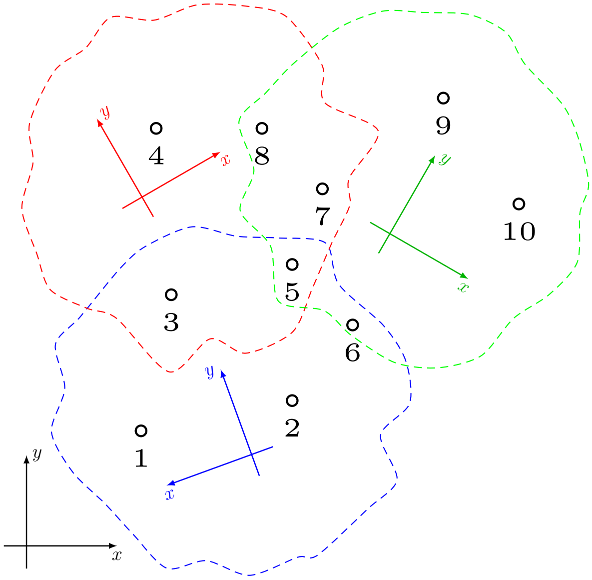

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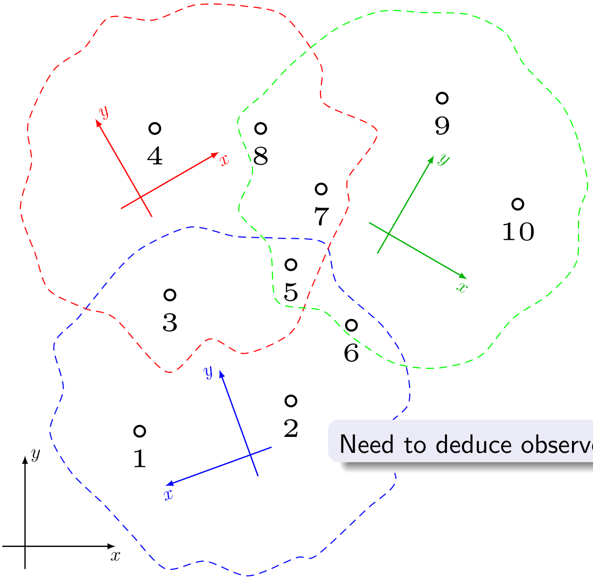

Need to deduce observers' transforms

System of equations

GIVEN

Nodes: 1, . . ., N (in $\mathbb{R}^d$)

Patches: P_1, . . ., P_M $\quad\quad \mathrm{P}_i \subset \{1, \ldots, N\}$

$\mathbf{x}_{k,i}$: local coordinate of node k if $k \in \mathrm{P}_i$

UNKNOWNS

$\mathbf{z}_k$: global coordinate of node k

$\mathcal{R}_i$: rigid transform corresponding to P_i, i.e. if $k \in \mathrm{P}_i$

$$\mathbf{z}_k = \mathcal{R}_i(\mathbf{x}_{k,i}) = \mathbf{O}_i\, \mathbf{x}_{k,i} + \mathbf{t}_i$$

Registration Problem

Find $\mathbf{z}_1, \ldots, \mathbf{z}_\mathrm{N}$, $\mathcal{R}_1, \ldots, \mathcal{R}_\mathrm{M}$ such that

$$\mathbf{z}_k = \mathcal{R}_i(\mathbf{x}_{k,i}), \quad k \in \mathrm{P}_i, \quad i \in [1:\mathrm{M}]. \qquad (\textsc{reg})$$

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$$\begin{aligned} \mathcal{R} : \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ \mathbf{x} &\longmapsto \mathbf{O}\mathbf{x} + \mathbf{t} \end{aligned}$$

$\mathbf{O}$: orthogonal matrix, $\mathbf{t}$: translation

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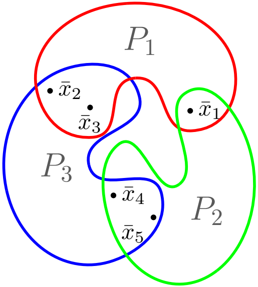
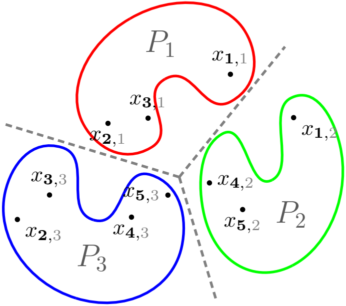


Original network



Given data

Registration Problem

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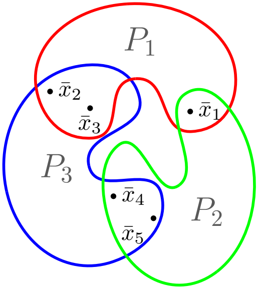

Original network

	observer 1	observer 2	observer 3	GLOBAL
node 1	$\mathbf{x}_{1,1}$	$\mathbf{x}_{1,2}$		$\mathbf{z}_1$
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node 5		$\mathbf{x}_{5,2}$	$\mathbf{x}_{5,3}$	$\mathbf{z}_5$

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$$k \in \mathrm{P}_i, \quad i \in [1 : \mathrm{M}]. \qquad \text{(REG)}$$

$$\mathbf{z}_1 = \mathcal{R}_1(\mathbf{x}_{1,1}) = \mathcal{R}_2(\mathbf{x}_{1,2})$$
$$\mathbf{z}_2 = \mathcal{R}_1(\mathbf{x}_{2,1}) = \mathcal{R}_3(\mathbf{x}_{2,3})$$
$$\mathbf{z}_3 = \mathcal{R}_1(\mathbf{x}_{3,1}) = \mathcal{R}_3(\mathbf{x}_{3,3})$$
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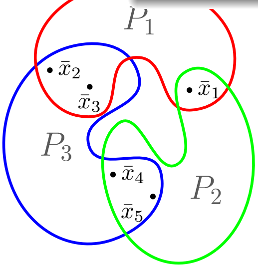

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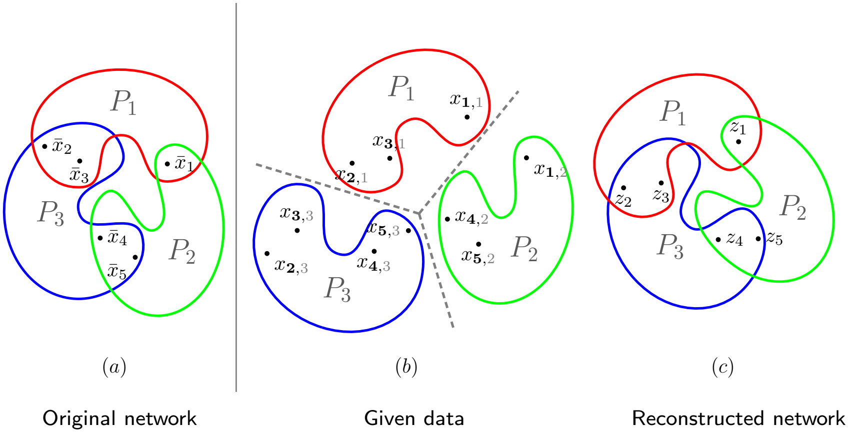
- Does a solution exist?

  Yes! Ground truth

- Is this solution unique ... up to congruence?

  We are interested only in relative positions and transformations



P_1
$\bar{x}_2$
$\bar{x}_3$
$\bar{x}_1$
P_3
$\bar{x}_4$
$\bar{x}_5$
P_2
(a)
P_1
$x_{\mathbf{1},1}$
$x_{\mathbf{3},1}$
$x_{\mathbf{2},1}$
$x_{\mathbf{1},2}$
$x_{\mathbf{3},3}$
$x_{\mathbf{5},3}$
$x_{\mathbf{4},2}$
$x_{\mathbf{2},3}$
$x_{\mathbf{4},3}$
$x_{\mathbf{5},2}$
P_2
P_3
(b)
P_1
z_1
z_2
z_3
P_2
P_3
z_4
z_5
(c)
Original network
Given data
Reconstructed network

Theorem: Uniqueness of solution

Suppose, for $\textsc{reg}$ in $\mathbb{R}^d$

A1. each patch contains at least $d + 1$ nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to $\textsc{reg}$ $\equiv$ rigidity of the body graph

Theorem: Uniqueness of solution

Rationale

Rigid transform determined by action on $d+1$ points in generic positions

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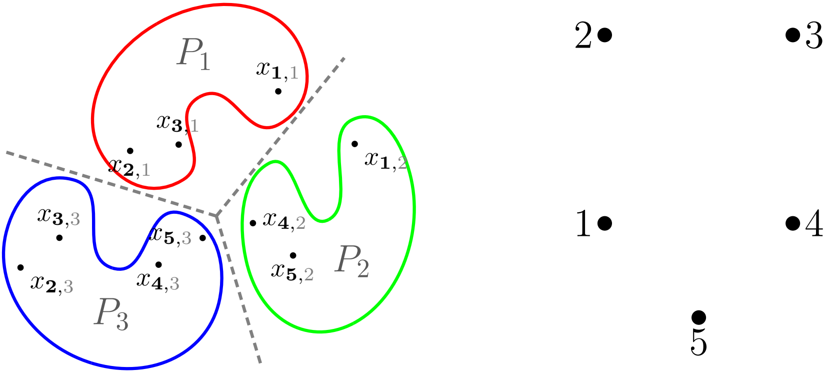
uniqueness of solution to REG ≡ rigidity of the **body graph**



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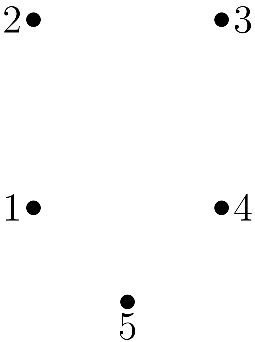

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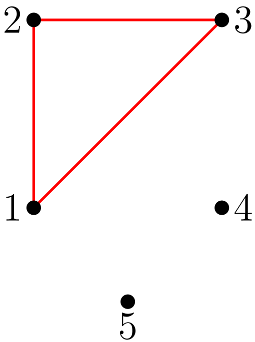

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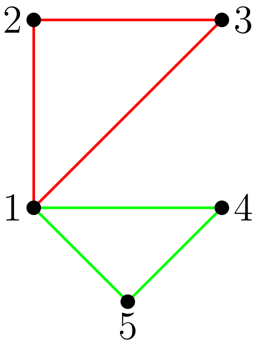

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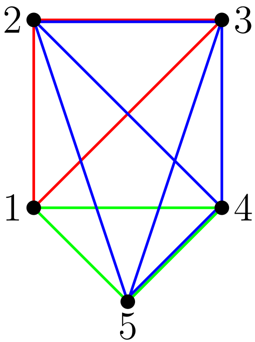

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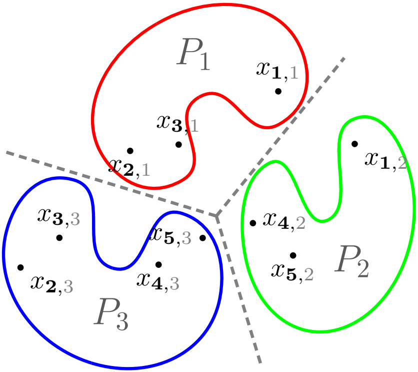
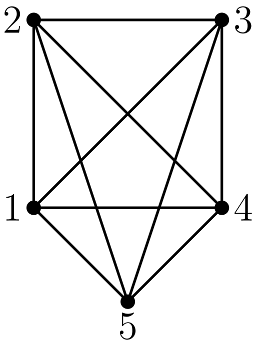


To test if this can be uniquely registered ...



... test if this graph is rigid

Graph (embedding) rigidity: Setup

GIVEN

- undirected graph: $\mathrm{G} = (V, E)$
- embedding of G in $\mathbb{R}^d$: mapping $V \to \mathbb{R}^d$

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$$\mathrm{G} = \Big(\underbrace{\{1,2,3\}}_{V}, \underbrace{\{(1,2),(1,3),(2,3)\}}_{E}\Big)$$

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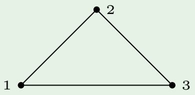
Embedding in $\mathbb{R}^2$



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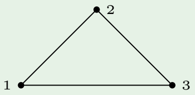

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QUESTION: Can we have an embedding which preserves edge lengths, but has a different shape?

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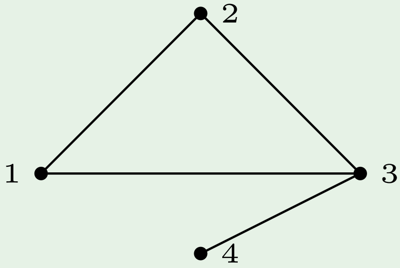

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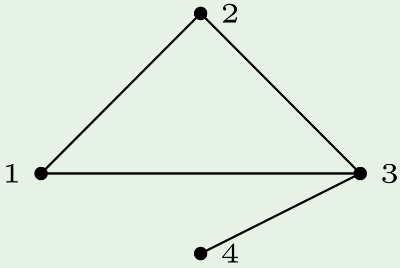

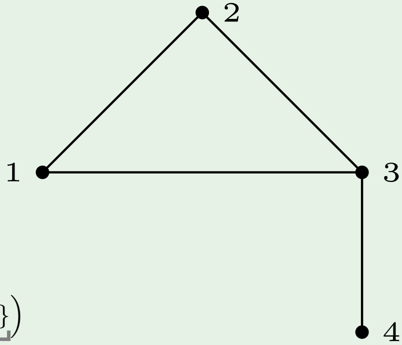

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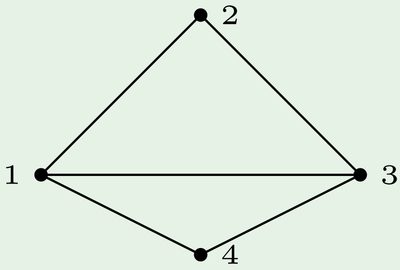

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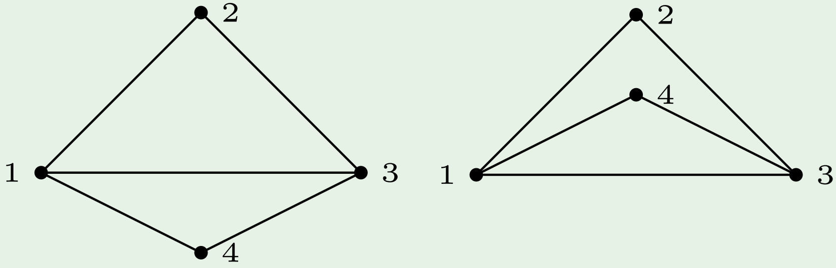

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Graph (embedding) rigidity: Graph vs Embedding

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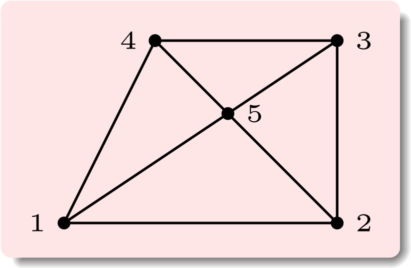

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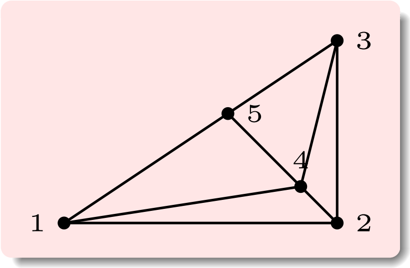

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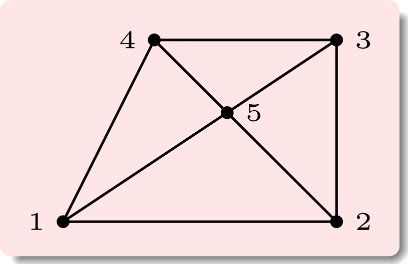

This embedding is **not rigid**

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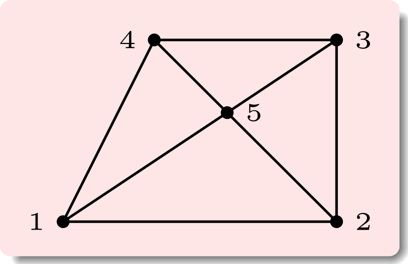

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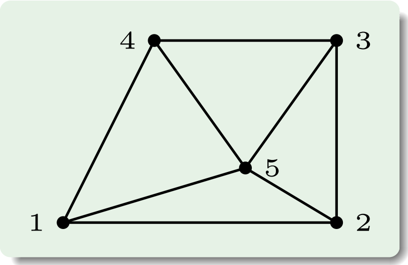

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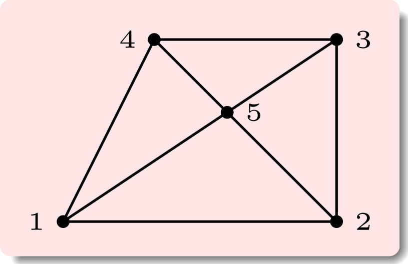

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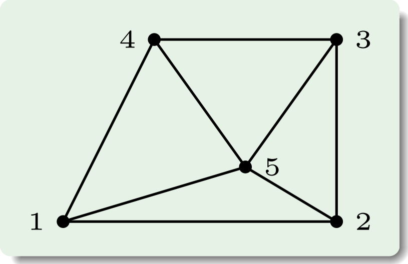

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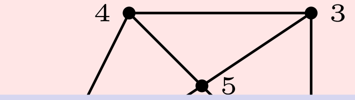

But recall our theorem . . .

Suppose, for REG in $\mathbb{R}^d$

A1. each patch contains at least $d+1$ nodes
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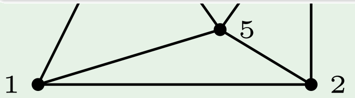
Then

uniqueness of solution to REG $\equiv$ rigidity of the **body graph**



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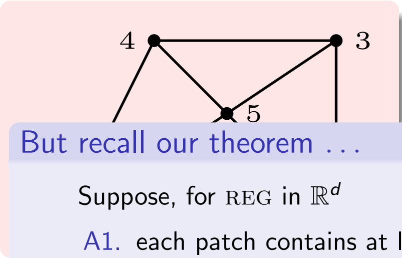

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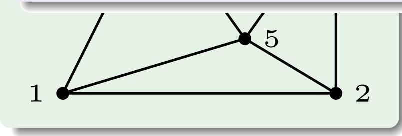
uniqueness of solution to $\textsc{reg}$ $\equiv$ rigidity of the **body graph**



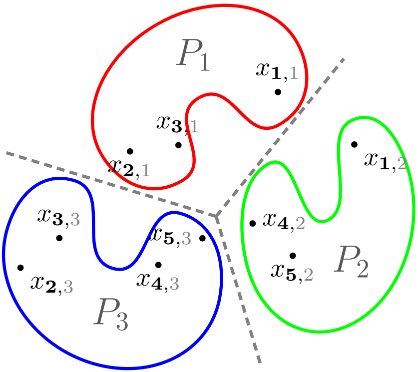
Graph ~~(embedding)~~ rigidity: Generic embedding

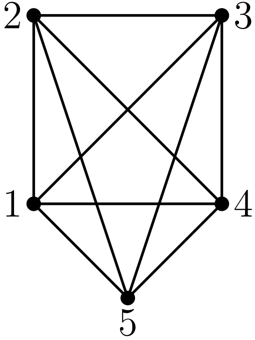
Rigidity is a generic property

Given a graph, **one** of the following is true

- **every** generic embedding is rigid
- **every** generic embedding is non-rigid

Generic embedding $\Longrightarrow$ rigidity becomes a **property of the graph**





To test if this can be uniquely registered . . .

. . . test if this graph is rigid

Theorem: Uniqueness of solution

Suppose, for $\textsc{reg}$ in $\mathbb{R}^d$

A1. each patch contains at least $d + 1$ nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to $\textsc{reg}$ $\equiv$ rigidity of the body graph

Corollary for $\mathrm{d} = 2$: $\textsc{reg}$ in $\mathbb{R}^2$

Suppose, in a two-dimensional network

A1. each patch contains at least 3 nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to $\textsc{reg}$ $\equiv$ 3-connectivity of the body graph

Corollary for $\mathrm{d} = 2$: $\textsc{reg}$ in $\mathbb{R}^2$

Suppose, in a two-dimensional network

A1. each patch contains at least 3 nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to $\textsc{reg} \equiv$ 3-connectivity of the body graph

connected graph

$\exists$ path between every pair of vertices

3-connected graph

remains connected if ≤ 3 vertices removed

Corollary for $d = 2$: REG in $\mathbb{R}^2$

Suppose, in a two-dimensional network

A1. each patch contains at least 3 nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to REG $\equiv$ 3-connectivity of the body graph

can be tested in linear time

3-connected graph

remains connected if ≤ 3 vertices removed

Corollary for $\mathrm{d} = 2$: $\textsc{reg}$ in $\mathbb{R}^2$

Suppose, in a two-dimensional network

A1. each patch contains at least 3 nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to $\textsc{reg} \equiv$ 3-connectivity of the body graph

can be tested in linear time

existing tests for 2D rigidity: quadratic time

Summary

- Registration problem: **assign global coordinates** to points based on partial observations in $\underbrace{\text{different local coordinate systems}}_{\textbf{related via rigid transforms}}$
- Focus: when is the solution unique
- Under mild assumptions: uniqueness equivalent to rigidity of the body graph
- Corollary for 2D networks: need only test 3-connectivity (linear time)

Thank You