Efficient Parameter Estimation for Semi-Continuous Data: An Application to Independent Component Analysis

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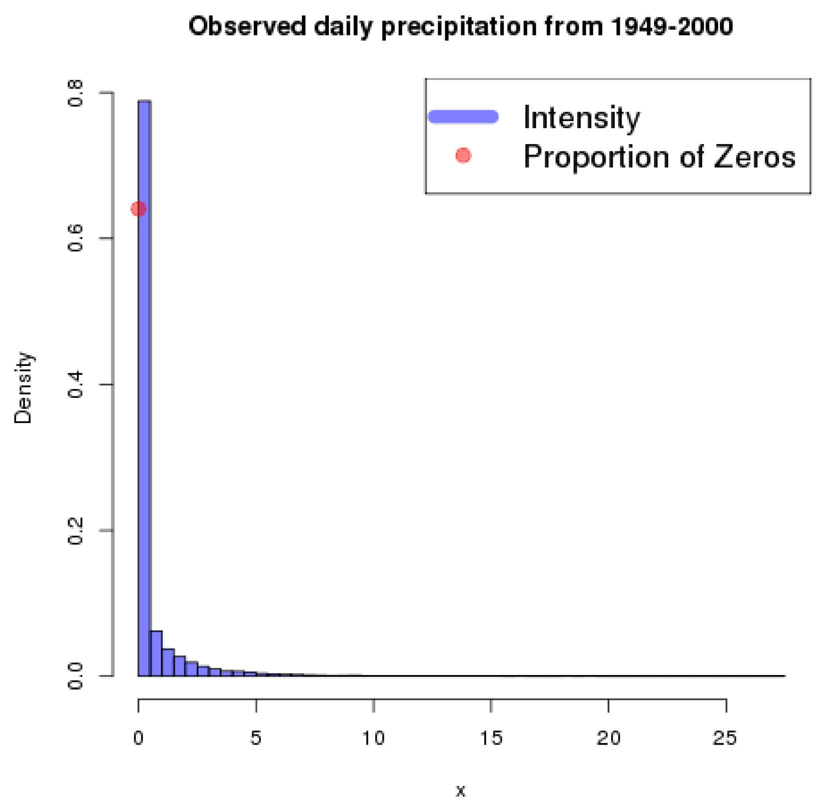
Introduction

Semi-continuous data have a point mass at zero and are continuous with positive support. Such data arise naturally in several real-life situations like daily rainfall at a location, sales of durable goods among many others. Therefore, efficient estimation of the underlying probability density function (PDF) is of significant interest.



Contribution

- We present an estimation method for semi-continuous data based on the maximum entropy principle.
- We demonstrate its successful application in developing a new Independent Component Analysis (ICA) algorithm, ICA-Semi-continuous Entropy Maximization (ICA-SCEM).
- We present a theoretical analysis of the proposed estimation technique and using simulated data we demonstrate the superior performance of ICA-SCEM over classical ICA algorithms.

Estimation using entropy maximization

The PDF of a semi-continuous random variable Y can be written as[1]

$$p(y \mid \gamma, \theta) = \gamma\delta(y) + (1-\gamma)\delta^*(y)g(y \mid \theta),$$

where $g(y \mid \theta)$ is a PDF of a continuous random variable with support on $(0, \infty)$, γ is the point mass at zero, $\delta(y)$ is the indicator function, and $\delta^*(y) = 1 - \delta(y)$.

Maximum Entropy Principle:

$$\max_{p(y)} H(p(y)) = -\int_I p(y) \log p(y)\, \mu(dy)$$

$$\text{s.t.} \int_I h_i(y)p(y)\, \mu(dy) = \alpha_i, \text{ for } i = 1, \ldots, K,$$

where $h_i(y)$ are measuring functions, $\alpha_i = \sum_{t=1}^{T} h_i(t)/T$ are the sample averages, and K denotes the total number of measuring functions.

Using the maximum entropy principle estimate the distribution that maximizes the entropy of Y.

For known γ, the distribution that maximizes the entropy of Y is given by[1]

$$p(y) = \gamma\delta(y) + (1-\gamma)\delta^*(y)g^*(y),$$

where g^* maximizes the entropy of a continuous random variable with support $(0, \infty)$ subject to the constraints $\int_0^\infty h_i(z)g(z)dz = \frac{\alpha_i}{1-\gamma}$, $i = 1, \ldots, K$.

[1] S. K. Popuri, "Prediction Methods for Semi-continuous Data with Applications in Climate Science," *Ph.D. thesis, University of Maryland, Baltimore County*, 2017.

Example

Suppose we have a sample of size n of semi-continuous data $\{y_1, \ldots, y_n\}$, γ is set to the proportion of zeroes in the data, and

$$\alpha_2 = \frac{1}{n}\sum_{i:y_i>0} y_i \text{ and } \alpha_3 = \frac{1}{n}\sum_{i:y_i>0} \log(y_i).$$

The resulting MaxEnt distribution is given by

$$f(y \mid \gamma, \kappa, \theta) = \gamma\delta(y) + (1-\gamma)\delta^*(y)\frac{y^{\kappa-1}e^{-y/\theta}}{\theta^\kappa\Gamma(\kappa)},$$

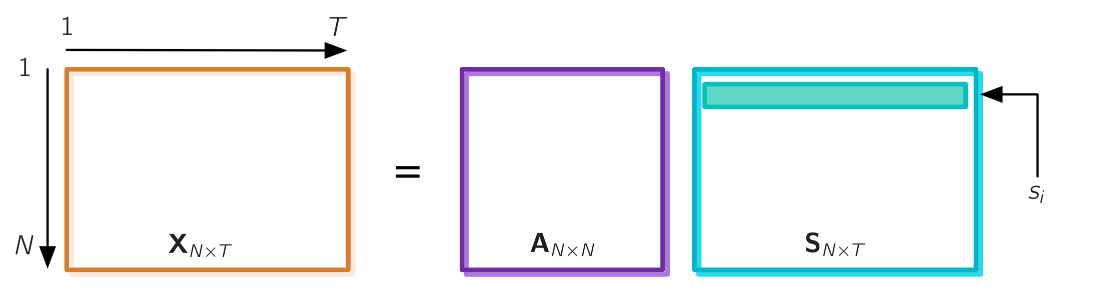
where θ and κ are solutions to

$$\alpha_2 = \kappa\theta \text{ and } \alpha_3 = \psi(\kappa) + \log(\theta),$$

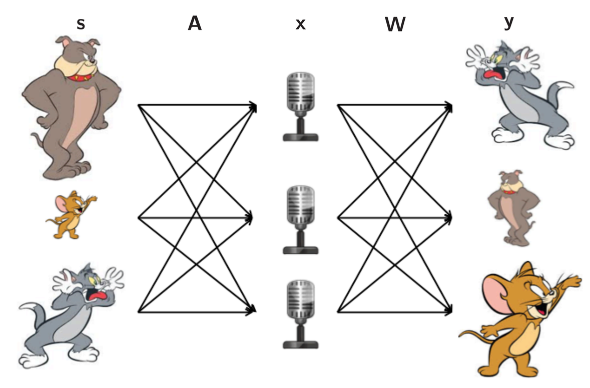
where $\psi(.)$ is the digamma function.

Application to ICA

Generative model: $\mathbf{x} = \mathbf{A}\mathbf{s}$, where $\mathbf{x}$ are the observations and $\mathbf{s}$ are the latent sources linearly mixed by matrix $\mathbf{A}$.



ICA can separate mixed sources subject to scaling and permutation ambiguities by assuming source *independence*.



In order to estimate $\mathbf{W}$, we *minimize* the mutual information (MI) among the source estimates[2] $y_1, \ldots, y_N$

$$J_{ICA}(\mathbf{W}) = \sum_{n=1}^{N} H(y_n) - \log|\det(\mathbf{W})| - H(\mathbf{x}),$$

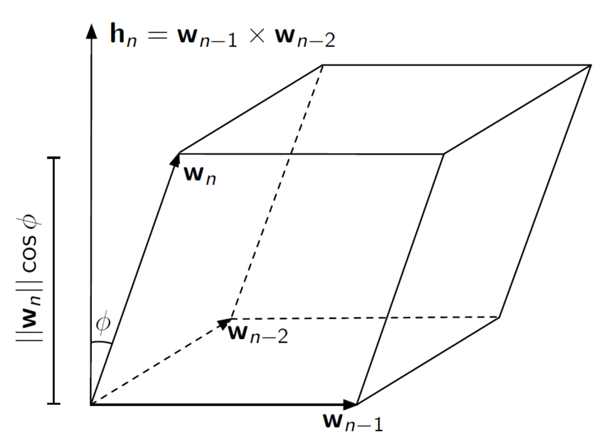
where $H(y_n) = -E\{\log(p(y_n))\}$.

Each y_n is assumed to be semi-continuous.

Development of ICA-SCEM algorithm

Decoupling the MI cost function enables for the development of effective algorithms[2].

This is achieved by expressing the volume of the parallelepiped, $|\det(\mathbf{W})|$, as the product of the area of its base and its height[3].



The cost function with respect to each $\mathbf{w}_n$ is given by

$$J_{ICA}(\mathbf{W}) = \sum_{n=1}^{N} H(y_n) - \log|(\mathbf{h}_n^\top \mathbf{w}_n)| - \log|\det(\mathbf{W}_n\mathbf{W}_n^\top)| - H(\mathbf{x}). \quad (1)$$

The gradient of (1) can be written in the decoupled form

$$\frac{\partial J(\mathbf{W})}{\partial \mathbf{w}_n} = -E\{\phi(y_n)\mathbf{x}\} - \frac{\mathbf{h}_n}{\mathbf{h}_n^\top \mathbf{w}_n}, \quad (2)$$

where $\phi(y_n) = \frac{\partial \log p(y_n)}{\partial y_n}$. As can be seen in (2), each gradient direction depends directly on the corresponding estimated source PDF and

$$\frac{\partial \log p(y_{n,t})}{\partial y_{n,t}} = \begin{cases} 0, & \text{if } y_{n,t} = 0 \\ \frac{\partial \log g(y_{n,t}|\theta_{n,t})}{\partial y_{n,t}}, & \text{if } y_{n,t} > 0. \end{cases}$$

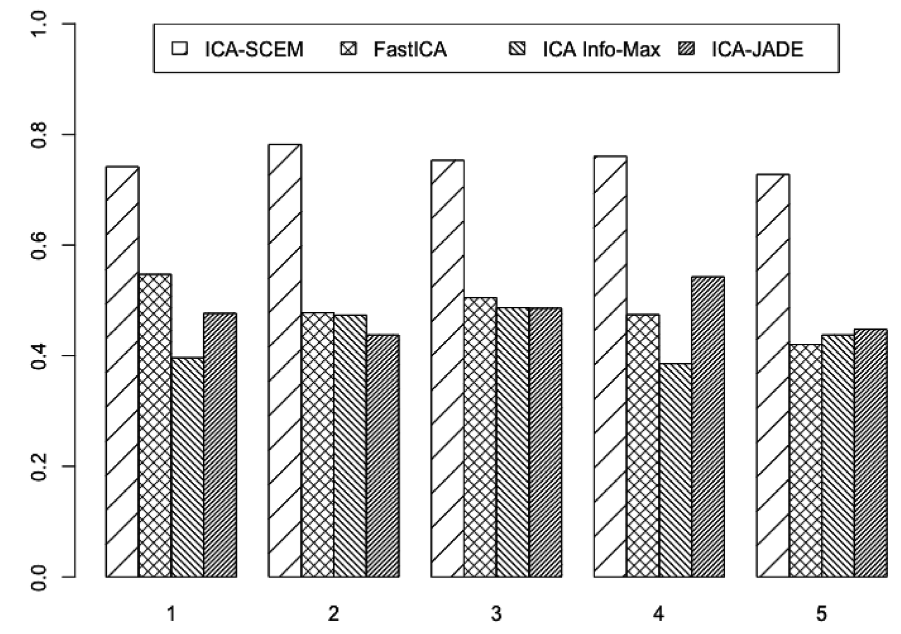
and $\phi(y_n) = [\frac{\partial \log p(y_{n,1})}{\partial y_{n,1}}, \ldots, \frac{\partial \log p(y_{n,T})}{\partial y_{n,T}}]^\top$ is a vector of partial derivatives of dimension T.

Experimental results

Simulation 1: Data for each source is generated using the two-part gamma distribution

$$f(y \mid \gamma, \kappa, \theta) = \gamma\delta(y) + (1-\gamma)\delta^*(y)\frac{y^{\kappa-1}e^{-y/\theta}}{\theta^\kappa\Gamma(\kappa)},$$

where $\gamma = 0.6$, $\theta = 1$, and $\kappa = 1$

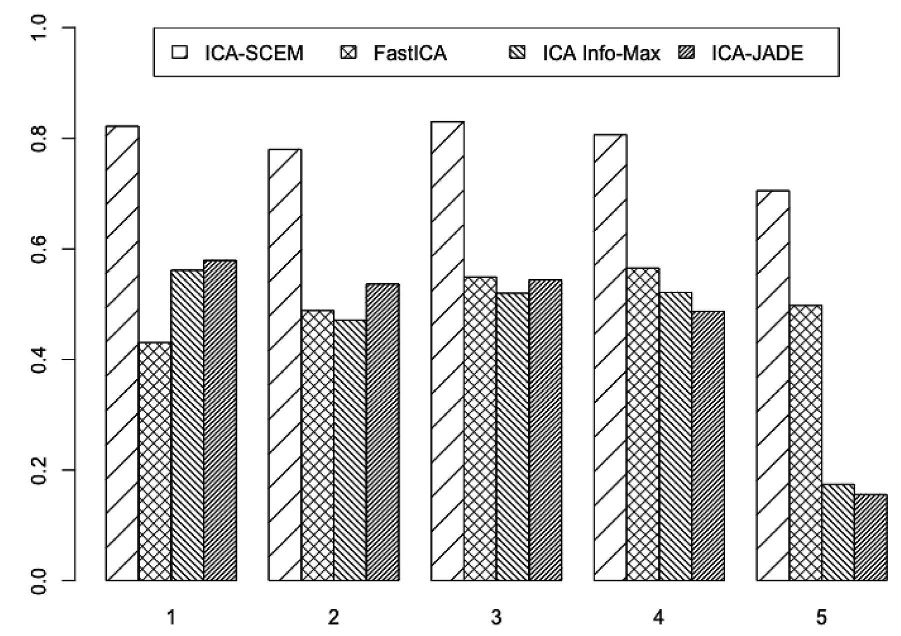


Simulation 2: Data for the first two out of five sources are generated using the two-part gamma model and for the rest of the three sources are generated using the following two-part lognormal distribution

$$f(y \mid \gamma, \mu, \sigma) = \gamma\delta(y) + (1-\gamma)\delta^*(y)\frac{1}{y}\phi\left(\frac{\log(y) - \mu}{\sigma}\right),$$

where data for the five sources are generated according the following parameter choices

Source	γ	κ	θ	μ	σ
1	0.6	1	1	–	–
2	0.4	1	2	–	–
3	0.6	–	–	0	1
4	0.5	–	–	0.5	0.5
5	0.4	–	–	1	2



ICA-SCEM performs the best among well known ICA algorithms in terms of separation performance

Conclusion and future directions

An efficient density estimation method for semi-continuous data was presented and a new ICA algorithm for semi-continuous data, ICA-SCEM, is proposed.

Future Directions:

- Comparisons of ICA-SCEM, with ICA algorithms that exploit the sparsity of the data as well as non-negative source separation based methods.
- Multivariate extensions could be developed by considering multivariate distributions for the continuous part with element-wise Bernoulli probabilities determining the presence of zeros.

[2] T. Adalı, M. Anderson, and G.-S. Fu, "Diversity in independent component and vector analyses: Identifiability, algorithms, and applications in medical imaging," *IEEE Signal Processing Magazine*, vol. 31, no. 3, pp. 18-33, May 2014.

[3] Z. Boukouvalas, Y. Levin-Schwartz, R. Mowakeaa, G.-S. Fu, and T. Adalı, "Independent Component Analysis Using Semi-Parametric Density Estimation Via Entropy Maximization," *In 2018 IEEE Statistical Signal Processing Workshop (SSP)*, pp. 403-407, 2018.