The Graph FRI Framework–Spline Wavelet Theory and Sampling on Circulant Graphs

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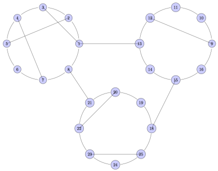

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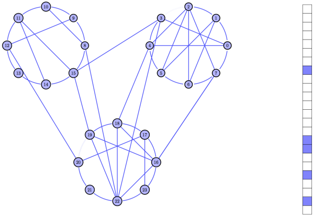

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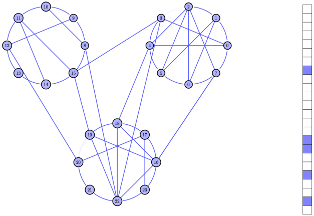

- Explore sparsity on circulant graphs, while demonstrating intuitive and concrete links to SP in the Euclidean domain
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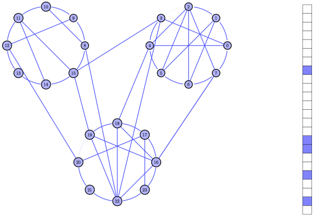
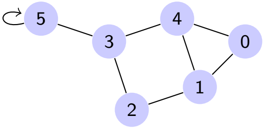


- Explore sparsity on circulant graphs, while demonstrating intuitive and concrete links to SP in the Euclidean domain
- Circulant graphs are LSI & circulant matrices are diagonalizable by the DFT-matrix
- **Wavelet Analysis:** Circulant Graph (E-)Spline Wavelet Transforms with reproduction & annihilation properties
- **Sparse Sampling:** Perfect recovery of sampled (wavelet-)sparse graph signals on coarsened graphs
- We can generalize operations to **arbitrary graphs** via approximation schemes

Signal Processing on Graphs

- A graph $G = (V, E)$, with $|V| = N$, is described via a vertex set $V = \{V_0, \ldots, V_{N-1}\}$, and an edge set $E = \{E_0, \ldots, E_{M-1}\}$



- The connectedness of G is represented by adjacency matrix $\mathbf{A}$, where

$$\mathbf{A}_{i,j} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent} \quad (i \neq j) \\ 0, & \text{otherwise} \end{cases}$$

- The degree matrix $\mathbf{D}$ of G is given by

$$\mathbf{D}_{i,j} = \begin{cases} \sum_j \mathbf{A}_{i,j}, & \text{if} \quad i = j \\ 0, & \text{otherwise} \end{cases}$$

- The non-normalized graph Laplacian is given by $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- We consider undirected, and (un-)weighted connected graphs without self-loops

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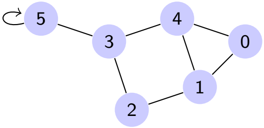

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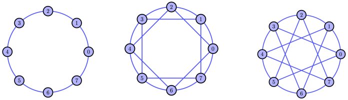
- The non-normalized graph Laplacian is given by $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- We consider undirected, and (un-)weighted connected graphs without self-loops
- A graph signal is a scalar function $\mathbf{x} : V \rightarrow \mathbb{C}$ defined on G such that $x(V)$ is the sample value of $\mathbf{x} \in \mathbb{C}^N$ at vertex V
- $\mathbf{L}$ has a complete set of orthonormal eigenvectors $\{\mathbf{u}_l\}_{l=0,\dots,N-1}$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{N-1}$.
- The Graph Fourier Transform (GFT) of a graph signal $\mathbf{x}$ on G is given by $\mathbf{X}^G = \mathbf{U}^H \mathbf{x}$

Circulant Graph Theory

Definition

- A graph G is circulant with respect to a generating set $S = \{s_1, \dots, s_M\}$, with $0 < s_k \leq N/2$ if there exists an edge between nodes $(i, (i \pm s_k)_N)$, for every $s_k \in S$. Alternatively, a graph is circulant if its graph Laplacian matrix $\mathbf{L}$ is circulant.

- We can express $\mathbf{L}$ with first row $[l_0 \quad \dots \quad l_{N-1}]$, via its representer polynomial $l(z) = \sum_{i=0}^{N-1} l_i z^i$, whereby $\mathbf{L} = \sum_{i=0}^{N-1} l_i \mathbf{\Pi}^i$ for circulant permutation matrix $\mathbf{\Pi}$.



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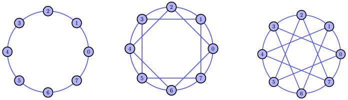

- The GFT of a circulant graph can be represented as a permutation of the DFT matrix
- Prior research [Ekambaram et al, '13]: fundamental SP operations on circulant graphs, incl. sampling & reconnection strategies, multiscale wavelet analysis (*spline-like graph wavelet filterbank*)

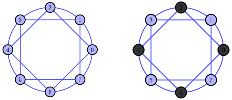

Figure 1: Downsampling w.r.t. $s = 1$ for $S = \{1, 2\}$

Prior Work: Graph Spline Wavelet Transform

- **Lemma 1.** For an undirected, circulant graph $G = (V, E)$, with $|V| = N$, the representer polynomial $l(z) = l_0 + \sum_{i=1}^{M} l_i(z^i + z^{-i})$ of graph Laplacian $\mathbf{L}$, with first row $[l_0\ l_1\ l_2\ \ \dots\ \ l_2\ l_1]$, has 2 **vanishing moments**. Thus, $\mathbf{L}$ annihilates up to linear polynomial graph signals, subject to a border effect determined by bandwidth M of $\mathbf{L}$, provided $2M < N$.

M. S. Kotzagiannidis, P. L. Dragotti, "Higher-order graph wavelets and sparsity on circulant graphs", *Proc. SPIE*, Wavelets and Sparsity XVI, 2015

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Theorem 1

- The higher-order graph-spline wavelet transform (HGSWT), for an undirected, connected circulant graph G, with adjacency matrix $\mathbf{A}$ and degree d per node, composed of the low-and high-pass filters

$$\mathbf{H}_{LP} = \frac{1}{2^k}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k \tag{1}$$

$$\mathbf{H}_{HP} = \frac{1}{2^k}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k = \frac{\mathbf{L}^k}{(2d)^k} \tag{2}$$

with $2k$ vanishing moments, is invertible as long as at least one node retains the lowpass component.

Proof: We can demonstrate that the nullspace of

$$\frac{1}{2^k}\left(\sum_{j\in\mathbb{Z}}\binom{k}{2j}\left(\frac{\mathbf{A}}{d}\right)^{2j} + \mathbf{K}\sum_{j\in\mathbb{Z}}\binom{k}{2j+1}\left(\frac{\mathbf{A}}{d}\right)^{2j+1}\right)$$

with downsampling pattern $\mathbf{K}$, is empty.

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A Generalized Graph Laplacian

Definition

- Let $G = (V, E)$, $|V| = N$, be undirected & circulant with adjacency matrix $\mathbf{A}$ of bandwidth M and degree $d = \sum_{k=1}^{M} 2d_k$ per node with symmetric weights $d_k = A_{i,(k+i)_N}$. Define the parameterised *e-graph Laplacian* of G as $\tilde{\mathbf{L}}_\alpha = \tilde{\mathbf{D}}_\alpha - \mathbf{A}$, where $\tilde{d}_\alpha = \sum_{k=1}^{M} 2d_k \cos(\alpha k)$ is the exponential degree.

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- Note that $\tilde{\mathbf{L}}_\alpha = \mathbf{L}$ for $\alpha = 0$
- When $\alpha = \frac{2\pi j}{N}$, we have that $\tilde{d}_\alpha = \lambda_j(\mathbf{A})$ for $j \in [0 \;\; N-1]$

$\Rightarrow$ we can interpret $\tilde{\mathbf{L}}_\alpha = \lambda_j \mathbf{I}_N - \mathbf{A}$ as a shift of $\mathbf{L}$ by $-\tilde{\lambda}_j(\mathbf{L}) = (\lambda_j - d)$ toward annihilation of $\mathbf{u}_j$:

$$(\lambda_j \mathbf{I}_N - \mathbf{A})\mathbf{u}_j = \mathbf{0}_N$$

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Lemma 2

- For undirected, circulant $G = (V, E)$, the representer polynomial $\tilde{l}(z)$ of $\tilde{\mathbf{L}}_\alpha$ has 2 **vanishing exponential moments**, i.e. $\tilde{\mathbf{L}}_\alpha$ annihilates complex exponential polynomial graph signals $y(t) = p(t)e^{\pm i\alpha t}$, $\alpha \in \mathbb{R}$ and $degp(t) = 0$. Unless $\alpha = \frac{2\pi k}{N}$, $\;k \in [0 \;\; N-1]$, this is subject to a border effect determined by M, whereby $2M < N$.

Proof: $\tilde{l}(z)$ factors $(1 - e^{i\alpha} z^{-1})(1 - e^{-i\alpha} z^{-1})$, which corresponds to 2 vanishing exponential moments.

Higher-Order Graph E-Spline Wavelet Transform

- Given connected, undirected circulant G, we create *higher-order graph e-spline wavelet transforms* for multiple parameters $\vec{\alpha} = (\alpha_1, \ldots, \alpha_T)$, composed of low-& high-pass filters of the form

$$\mathbf{H}_{LP\,\vec{\alpha}} = \prod_{n=1}^{T} \frac{1}{2^k} \left(\beta_n \mathbf{I}_N + \frac{\mathbf{A}}{d} \right)^k \tag{3}$$

$$\mathbf{H}_{HP\,\vec{\alpha}} = \prod_{n=1}^{T} \frac{1}{2^k} \left(\beta_n \mathbf{I}_N - \frac{\mathbf{A}}{d} \right)^k = \prod_{n=1}^{T} \frac{\mathbf{L}_{\alpha_n}^k}{(2d)^k} \tag{4}$$

  whereby $\mathbf{A}$ is the adjacency matrix, d the degree per node and parameter $\boxed{\beta_n = \frac{\tilde{d}_{\alpha_n}}{d}}$, with exponential degree $\tilde{d}_{\alpha_n} = \sum_{k=1}^{M} 2d_k \cos(\alpha_n k)$.

- $\mathbf{H}_{HP\,\vec{\alpha}}$ annihilates complex exponential polynomials (of $\deg p(t) \leq k-1$) with exponent $\pm i\alpha_n$

- The transform is invertible subject to restrictions on parameters k, T and β_n, as well as on the downsampling pattern.

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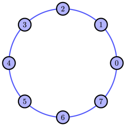
- The rows/columns of $\mathbf{H}_{LP_\alpha} = \frac{1}{2d}(2\tilde{\mathbf{D}}_\alpha - \tilde{\mathbf{L}}_\alpha)$ for $k = 1$ produce the discrete linear spline for $\alpha = 0$ & e-spline of order 2 for $\alpha \neq 0$



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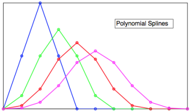

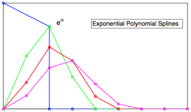

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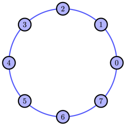

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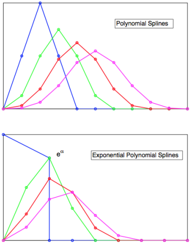

- **Corollary 1**: If $G = (V, E)$ is undirected, connected and bipartite circulant, the rows/columns of $\mathbf{H}_{LP_\alpha}$ can reproduce complex exponentials (*linear polynomials*)

$\Rightarrow$ Undirected, bipartite circulant graphs bear a spline-property: graph-splines & associated filterbanks with 2 vanishing (exponential) moments

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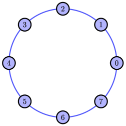

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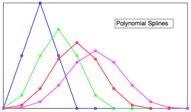

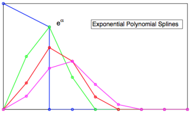

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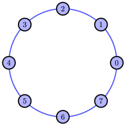

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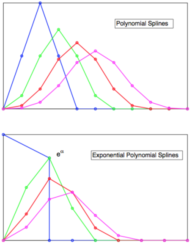

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- In general, the *HG(E)SWT*, whose high-pass filter $\mathbf{H}^k_{HP_\alpha}$ has $2k$ vanishing exponential moments, gives rise to higher-order graph-(e-)spline functions
- **Note:** Splines are traditionally defined via the Green's functions of a continuous differential operator

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- Design new graph wavelets, with well-defined synthesis filters, and reproduction properties, via **spectral factorization**

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- Given high-pass filter $H_{HP_{\vec{\alpha}}}(z)=\prod_{n=1}^{T}\frac{\tilde{l}_{\alpha_n}(z)^k}{(2d)^k}$, with $\tilde{H}_{LP_{\vec{\alpha}}}(z)=H_{HP_{\vec{\alpha}}}(-z)$, we derive *symmetric* analysis low-pass filter $H_{LP_{\vec{\alpha}}}(z)$: let $\boxed{P(z)=H_{LP_{\vec{\alpha}}}(z)\tilde{H}_{LP_{\vec{\alpha}}}(z)}$, subject to $\boxed{P(z)+P(-z)=2}$
- Can impose further constraints such as $H_{LP_{\vec{\alpha}}}(z)=\prod_{n=1}^{T}(z+2\cos(\alpha_n)+z^{-1})^k R(z)$ for unknown $R(z)$

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- Can impose further constraints such as $H_{LP_{\vec{\alpha}}}(z) = \prod_{n=1}^{T}(z + 2\cos(\alpha_n) + z^{-1})^k R(z)$ for unknown $R(z)$

Theorem 2

For undirected connected circulant graph $G = (V, E)$, with adjacency matrix $\mathbf{A}$ and node degree d, we define the *higher-order 'complementary' graph e-spline wavelet transform* (HCGESWT) via the analysis filters:

$$\mathbf{H}_{LP_{\vec{\alpha},an}} \stackrel{(*)}{=} \mathbf{C}\bar{\mathbf{H}}_{LP,\vec{\alpha}} = \mathbf{C}\prod_{n=1}^{T} \frac{1}{2^k}\left(\beta_n \mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k \tag{5}$$

$$\mathbf{H}_{HP_{\vec{\alpha},an}} = \prod_{n=1}^{T} \frac{1}{2^k}\left(\beta_n \mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k \tag{6}$$

and synthesis filters:

$$\mathbf{H}_{LP_{\vec{\alpha},syn}} = \mathbf{H}_{HP_{\vec{\alpha},an}} \circ I_{HP} \tag{7}$$

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where $\mathbf{H}_{LP_{\vec{\alpha},an}}$ is the solution to the above linear system, with circulant $\mathbf{C} = \mathbf{H}_{LP_{\vec{\alpha},an}}\bar{\mathbf{H}}_{LP,\vec{\alpha}}^{-1}$, and circulant indicator matrices $I_{LP/HP}$ with first row $[1\ -1\ 1\ -1\ \ldots]$.

* applies when $\bar{\mathbf{H}}_{LP,\vec{\alpha}}$ is invertible

Sampling of Sparse and Compressible Graph Signals

- We can obtain a sparse (multiresolution) representation of a ‘compressible’ graph signal $\mathbf{x}$, which we term **wavelet**-K-**sparse**, using a suitable GWT

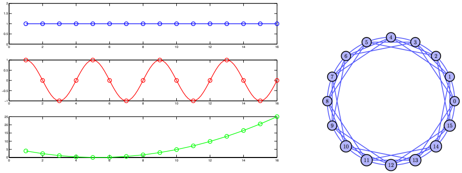

Figure 2: Compressible graph signals

- Iterate on the low-pass-branch and redefine the downsampled output on coarsened graphs

$\Rightarrow$ choose coarsening schemes with little to no reconnection, which preserve **circularity**

- Ultimately, we consider the multilevel representation of $\mathbf{x}$ on the initial graph G

The Graph-FRI Framework

- In discrete-time, a K-sparse signal $\mathbf{x} \in \mathbb{R}^N$ with measurement vector $\mathbf{y} = \mathbf{F}\mathbf{x}$, where $\mathbf{F} \in \mathbb{C}^{N \times N}$ is the DFT-matrix, can be perfectly reconstructed from $M \geq 2K$ consecutive sample values of $\mathbf{y}$ using *Prony's method*

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- We can sample and perfectly reconstruct a (wavelet-)K-sparse graph signal (with multiresolution) $\mathbf{x} \in \mathbb{C}^N$, on the vertices of circulant G using the dimensionality-reduced GFT representation $\mathbf{y} = \mathbf{U}_M^H \mathbf{x}$, $\mathbf{y} \in \mathbb{C}^M$, whereby $\mathbf{U}_M^H$ are the first M rows of $\mathbf{U}^H$, as long as $M \geq 2K$.

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} — & \mathbf{u}_1^H & — \\ — & \mathbf{u}_2^H & — \\ & \vdots & \\ — & \mathbf{u}_M^H & — \end{bmatrix}}_{GFT_M} \begin{bmatrix} | \\ | \\ \mathbf{x} \\ | \\ | \end{bmatrix}$$

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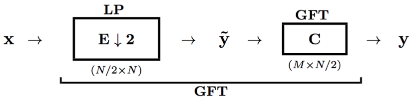
- But what graph is $\mathbf{y}$ associated with?

GFRI: Graph Coarsening



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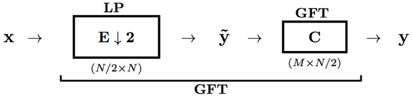

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Given GFT $\mathbf{y} \in \mathbb{C}^M$ from Thm 3., we determine the coarsened graph $\tilde{G} = (\tilde{V}, \tilde{E})$ associated with the dimensionality-reduced graph signal $\tilde{\mathbf{y}} \in \mathbb{C}^{\tilde{M}}$, via the decomposition

$$\mathbf{y} = \mathbf{U}_M^H \mathbf{x} = \mathbf{C} \prod_{j=0}^{J-1} (\mathbf{\Psi}_{j\downarrow 2} \mathbf{E}_{2^j \vec{\alpha}}) \mathbf{x} = \mathbf{C} \tilde{\mathbf{y}}$$

$\mathbf{U}_M^H \in \mathbb{C}^{M \times N}$: row-reduced permuted GFT basis
$\mathbf{C} \in \mathbb{C}^{M \times \tilde{M}}$: coefficient matrix with $\tilde{M} = \frac{N}{2^J}$,
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$\mathbf{E}_{2^j \vec{\alpha}} \in \mathbb{R}^{N/2^j \times N/2^j}$: (higher-order) graph e-spline low-pass filter on $\tilde{G}_j$, which reproduces complex exponentials at level j, with parameters $\vec{\alpha} = (\alpha_1, \ldots, \alpha_M) = \left(0, \ldots, \frac{2\pi(M-1)}{N}\right)$.

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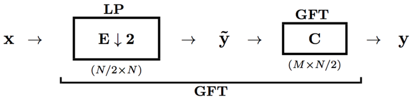

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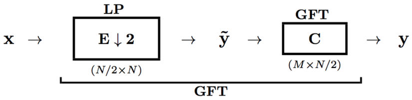

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GFRI: Graph Coarsening

- The coarsened graph $\tilde{G}_j$ at level j can be determined through two different schemes:

  (i) Using the pattern V_α in $\mathbf{\Psi}_{j\downarrow 2}$, perform Kron-reduction to obtain graph Laplacian $\mathbf{L}_j$ at level j

  $$\mathbf{L}_j = \mathbf{L}_{j-1}(V_\alpha, V_\alpha) - \mathbf{L}_{j-1}(V_\alpha, V_\alpha^C)\mathbf{L}_{j-1}(V_\alpha^C, V_\alpha^C)^{-1}\mathbf{L}_{j-1}(V_\alpha, V_\alpha^C)^T$$

  (ii) Define eigenbasis $(\tilde{\mathbf{U}}_j, \tilde{\mathbf{\Lambda}}_j) \in \mathbb{C}^{N/2^j \times N/2^j}$ at level $j \leq J$, through the projection of $\mathbf{\Psi}_{j-1\downarrow 2}$ on $(\tilde{\mathbf{U}}_{j-1}, \tilde{\mathbf{\Lambda}}_{j-1})$. The coarse graph $\tilde{G}_j$ for graph signal $\tilde{\mathbf{y}}_j = \prod_{k=0}^{j-1}(\mathbf{\Psi}_{k\downarrow 2}\mathbf{E}_{2^k\vec{\alpha}})\mathbf{x}$, has adjacency matrix

  $$\mathbf{A}_j = (2^j/N)\tilde{\mathbf{U}}_j\tilde{\mathbf{\Lambda}}_j\tilde{\mathbf{U}}_j^H$$

  which preserves generating set S of G for a sufficiently small bandwidth.

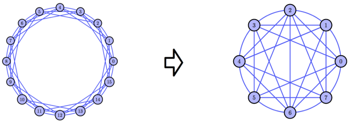

Figure 3: Scheme (ii) for a Graph with $S = \{1, 2, 3\}$

Generalizations to Arbitrary Graphs

Given an arbitrary graph G with adjacency matrix $\mathbf{A}$, we can resort to approximation schemes

- *Nearest Circulant Approximations*:

$$\tilde{\mathbf{A}} = \sum_{i=0}^{N-1} \frac{1}{N} \langle \mathbf{A}^P, \mathbf{\Pi}^i \rangle_F \mathbf{\Pi}^i, \quad \text{for circulant permutation matrix } \mathbf{\Pi}$$

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- *Graph Product Approximations*: impose circularity on factor graphs for multi-dimensional wavelet analysis

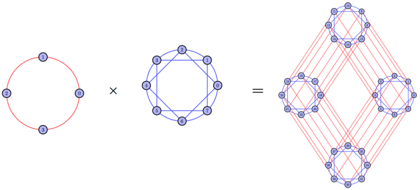

Figure 4: Cartesian Graph Product of Circulants

Conclusion and Future Work

- We have introduced a breadth of novel higher-order GWT constructions for the reproduction/annihilation of smooth graph signals
- By identifying arising spline-like functions, we can detect links to established concepts from classical SP
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- On that basis, we formulate a sampling and graph coarsening theory for sparse signals on graphs, extending the classical FRI framework
- Expand annihilation properties of existing (and/or evolved) GWT designs to more classes of graph signals

For a comprehensive discussion on circulant (e-)spline wavelets and extensions, refer to arXiv

Splines and Wavelets on Circulant Graphs
http://arxiv.org/abs/1603.04917

Thank you.

References I

- M.S. Kotzagiannidis, and P.L. Dragotti, "Sparse graph signal reconstruction and image processing on circulant graphs", *IEEE GlobalSIP*, pp. 923-927, 2014
- M. S. Kotzagiannidis and P. L. Dragotti, "Higher-order graph wavelets and sparsity on circulant graphs", in *Proc. SPIE*, Wavelets and Sparsity XVI, Vol. 9597 (SPIE, Bellingham, WA 2015), 95971E, 2015
- D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains", *SP Magazine, IEEE*, vol. 30, no. 3, pp. 83-98, 2013.
- Y. N. Ekambaram, G. Fanti, B. Ayazifar, and K. Ramchandran,"Circulant structures and graph signal processing", in *Proc. of the IEEE International Conference on Image Processing (ICIP)*, pp. 834-838, 2013.
- V. N. Ekambaram, G. Fanti, B. Ayazifar, and K. Ramchandran, "Critically-Sampled Perfect-Reconstruction Spline-Wavelet Filterbanks for Graph Signals", *IEEE GlobalSIP*, pp. 475-478, 2013.
- C. Vonesch, T. Blu, and M. Unser, "Generalized daubechies wavelet families", *IEEE Transactions on Signal Processing*, vol. 55, no. 9, pp. 4415-4429, Sept 2007.