WAVE PHYSICS INFORMED DICTIONARY LEARNING IN ONE DIMENSION

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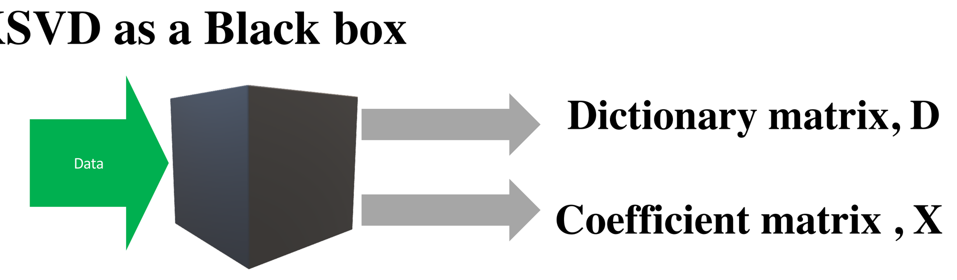
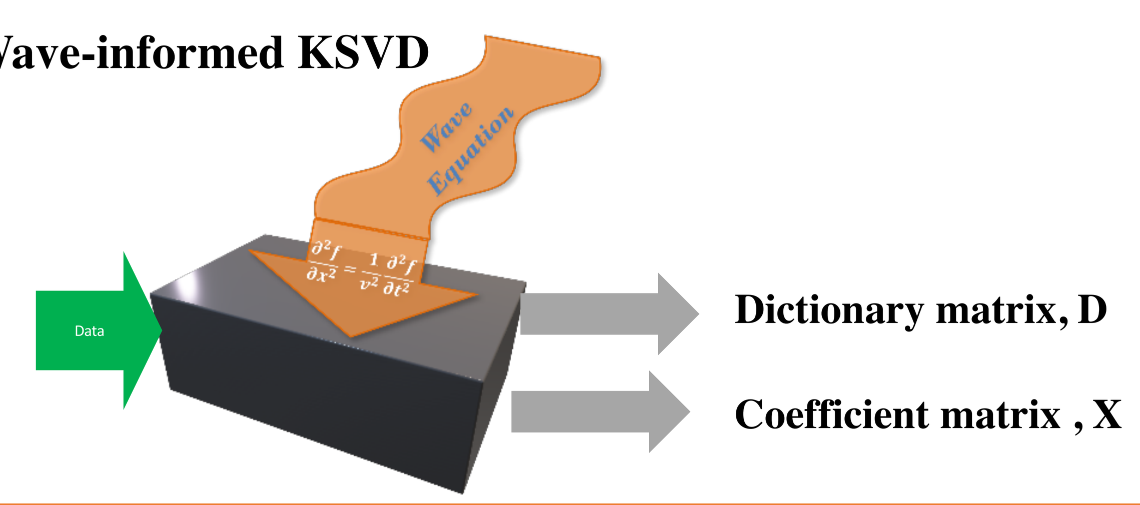
Introduction

- In Structural health monitoring, data-driven approaches to model behavior of waves to detect and locate damages has gained popularity.
- Recent works have used popular dictionary learning algorithm, K-SVD, to learn overcomplete dictionary for waves propagating in a metal plate.
- Instead of treating the K-SVD as a black box, we create a novel modification by enforcing domain knowledge.
- We look at how regularizing the K-SVD with one-dimensional wave equation affects the dictionary atoms

KSVD as a Black box



Wave-informed KSVD



Approach

The Physical Model: Domain Knowledge

- Wave Equation: $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$
- Space-time separability assumption:
  $$f(x,t) = d(x)q(t)$$
- Obtaining an eigenvalue problem by taking the Fourier transform (over time) of the wave equation with the space-time separability assumption:
  $$\frac{\partial^2 d}{\partial x^2}F\{q\{t\}\} = \frac{1}{v^2}d(x)F\left\{\frac{\partial^2 q(t)}{\partial t^2}\right\} \Rightarrow \frac{\partial^2 d}{\partial x^2} = \frac{-\omega^2}{v^2}d$$
- The discretized form of the Eigenvalue problem (to enforce physical consistence):
  $$\boldsymbol{LD_i} = g_k\boldsymbol{D}_\mathbf{i} \text{ where, } L_{3x3} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \text{(e.g.)}$$

Objective Function

$$\min_{\boldsymbol{D},\boldsymbol{X}} ||\boldsymbol{Y} - \boldsymbol{DX}||_F^2 + \sum_{j=1}^{K} \gamma_k || \boldsymbol{LD}_j - g_k\boldsymbol{D}_j ||_2^2$$

Optimization Approach

- Alternatively update $\boldsymbol{X}$ and $\boldsymbol{D}$ as shown in the algorithm block.
- Additionally the parameter g_k is also updated in each step and set $\gamma_k \propto \frac{1}{g_k^2}$.

Algorithm: Wave Informed K-SVD

Algorithm 1 wave-informed K-SVD, **Input**: $\mathbf{Y} \in R^{m\times n} K \in N$

Intialize $\mathbf{D}^{(0)}$, $\mathbf{g}^{(0)} = (g_1^{(0)}, g_2^{(0)}, \cdots, g_K^{(0)})$ and $iter$ (no. of iterations)
Set $t = 0$
repeat
  Sparse Code Stage:
  i = 1,2,...,N ; $\min_{\mathbf{X}_i}\{||\mathbf{Y}_i - \mathbf{D}^{(t)}\mathbf{X}_i||_F^2\}$ subject to $||\mathbf{X}_i||_0 \leq s$
  Dictionary Update Stage:
  $g_k^{(t)} = \mathbf{D}_k^{(t)T}\mathbf{L}\mathbf{D}_k^{(t)}; k = 1, 2, ..., K$
  $\mathbf{E}_k^{(t)} = \mathbf{Y} - \sum_{j\neq k}\mathbf{D}_j^{(t)}\widehat{\mathbf{X}}_j^{(t)T}; k = 1, 2, ..., K$
  Let S contain indices of columns that are non-zero. Now $\widetilde{\mathbf{E}}_k^{(t)}$ is formed from $\mathbf{E}_k^{(t)}$ by selecting columns indicated by S.
  Eigen Value Decomposition of $\widetilde{\mathbf{E}}_k^{(t)}\widetilde{\mathbf{E}}_k^{(t)T} - \gamma_k(\mathbf{L} - g_k\mathbf{I})(\mathbf{L} - g_k\mathbf{I})^T = \mathbf{U}\boldsymbol{\Delta}\mathbf{U}^{-1}$
  Choose column $\mathbf{D}_k^{(t)}$ to be first column of U
  Update $\widetilde{\mathbf{X}}_k^{(t)} = \widetilde{\mathbf{E}}_k^{(t)T}\mathbf{D}_k^{(t)}$
  $\widehat{\mathbf{X}}_k^{(t)}$ is constructed from $\widetilde{\mathbf{X}}_k^{(t)}$ by placing the elements of the latter at the indices indicated by S, zeros otherwise.
  $t \leftarrow t + 1$
until $t == iter$

Results

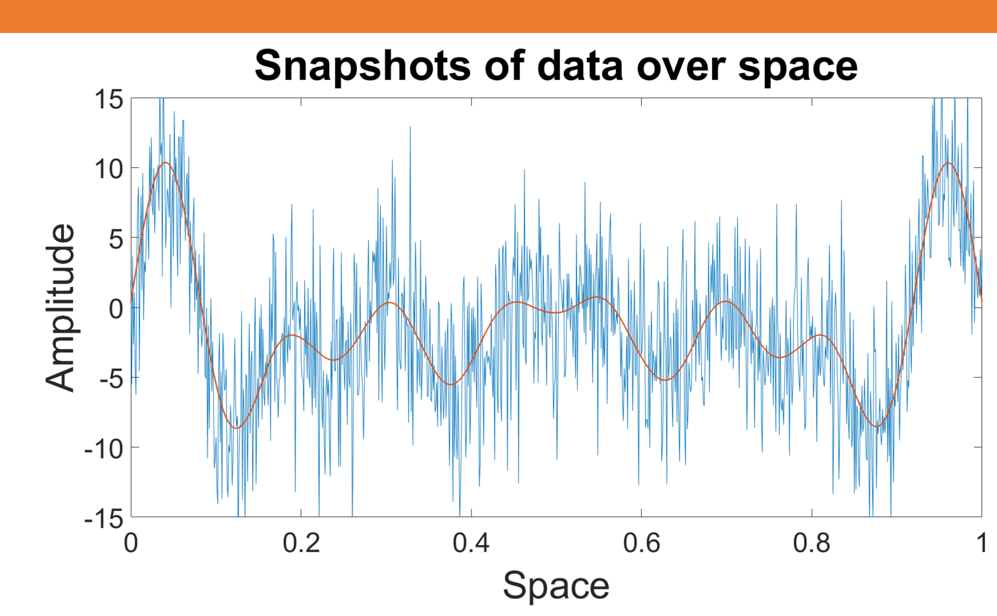
Data Model (fixed string):

$$y(x,t) = \sum_{k=1}^{8} a_k \sin((2k-1)x)\sin(\omega_k t + \phi_k)\, e^{-4t} + n(x,t)$$

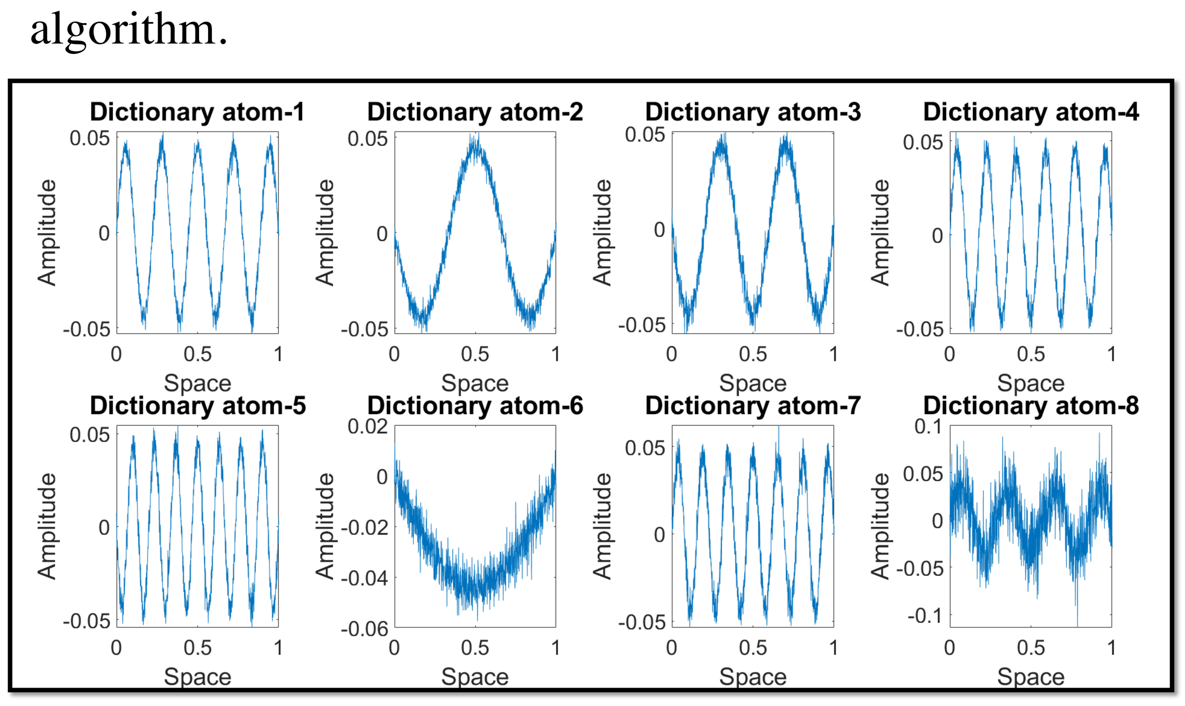
Where $\omega_k = vk$, where v is the velocity of the wave
And $n(x,t)$ is white Gaussian noise.

Algorithm Details:

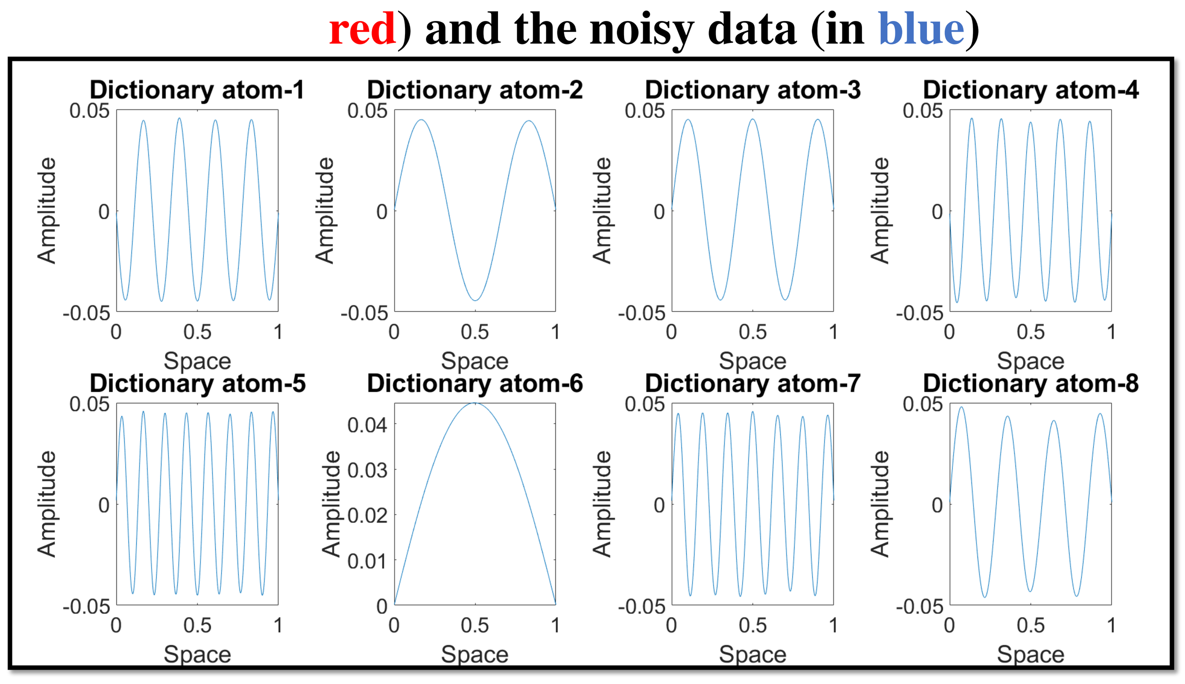
- $\gamma_0 \approx 10^7$, where $\gamma_k = \gamma_0/g_k^2$.
- Sparsity $s = 1$ in Orthogonal matching pursuit part of the algorithm.



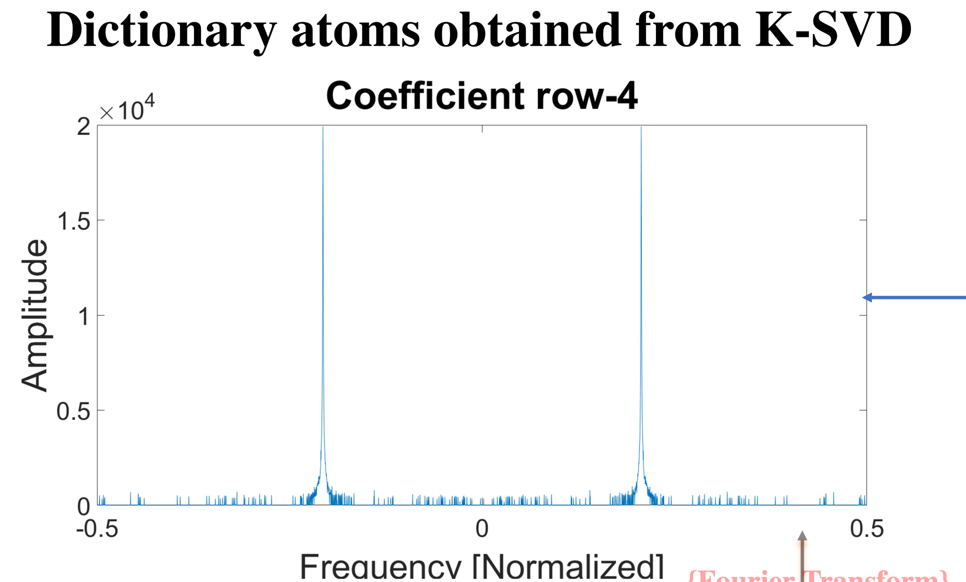
Data Reconstructed from Wave-Informed KSVD (in red) and the noisy data (in blue)

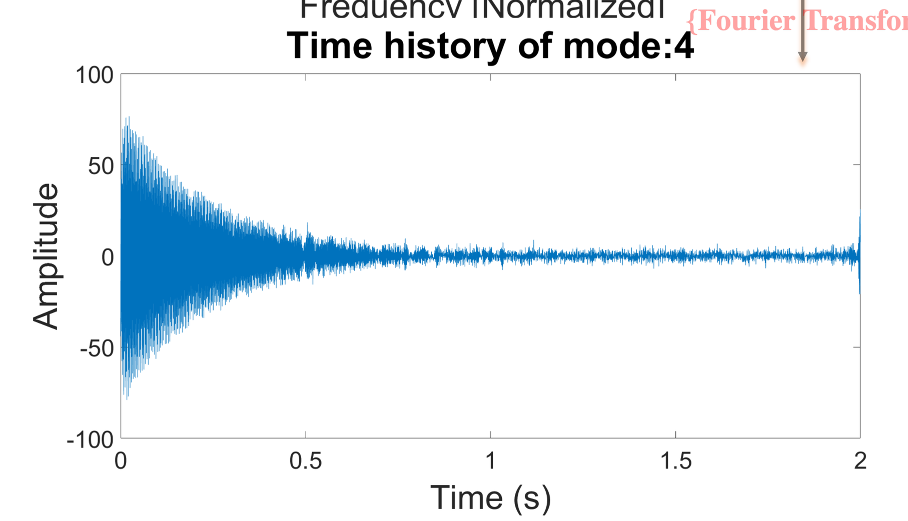


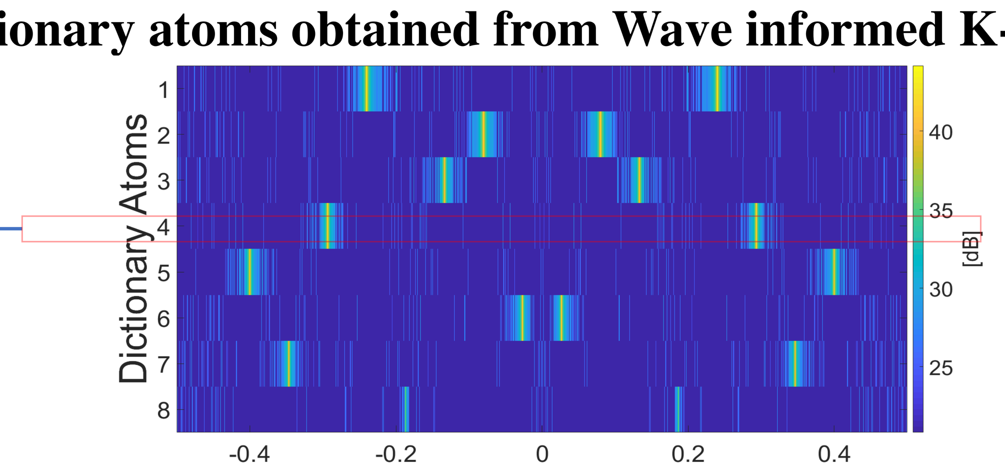
Dictionary atoms obtained from K-SVD

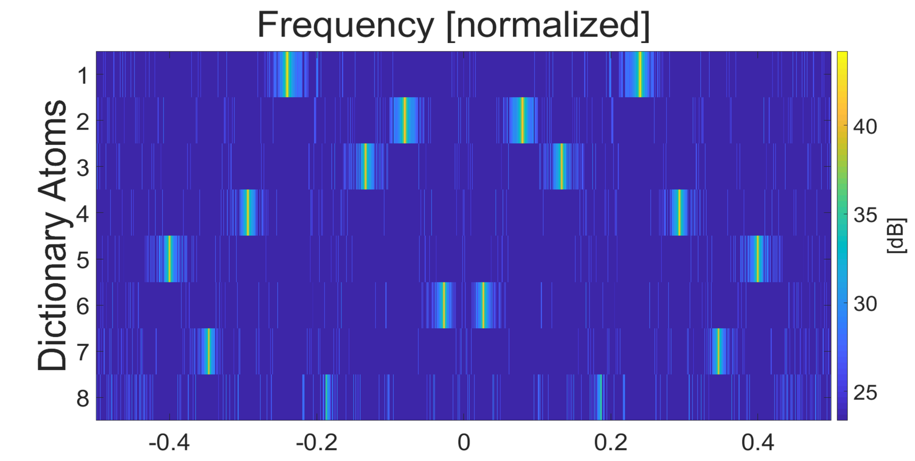


Dictionary atoms obtained from Wave informed K-SVD









The Coefficient Matrix from Wave-informed K-SVD (above) and K-SVD (below)

Conclusions

Observations and Future work

- Cleaner dictionary atoms from Wave-informed KSVD hints an enforcement of structure to the atoms.
- Future work will include modifying modern dictionary algorithms for wave data.
- Also, part of future work will include enforcing the wave constraint to sparse autoencoder

Acknowledgements

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