Sampling Signals on Meet/Join Lattices

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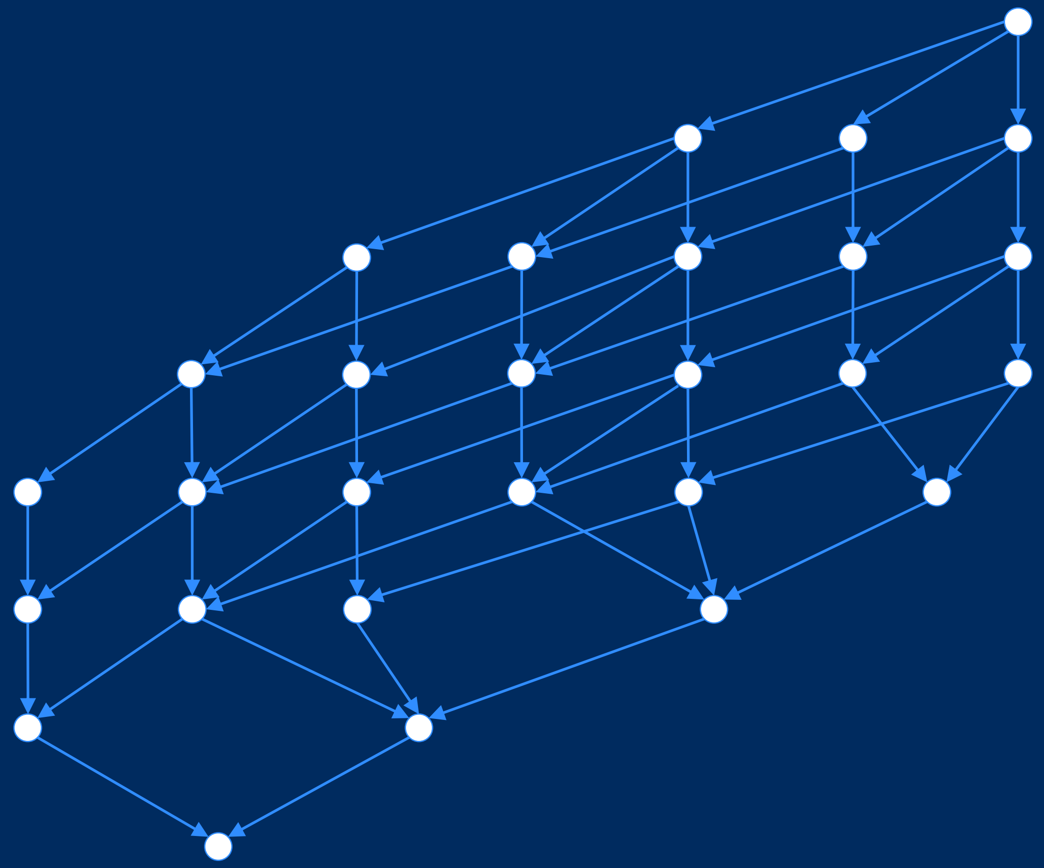
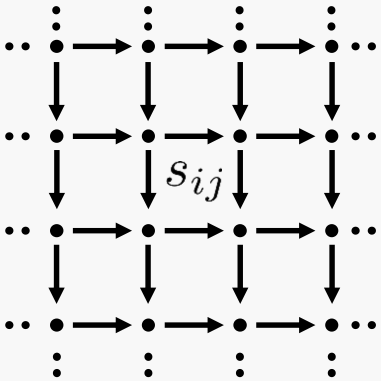


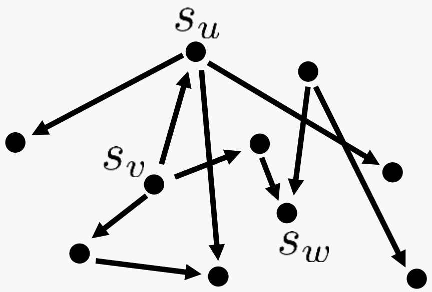
Computer Science
ETH zürich

Classical DSP



Signals indexed by time/space

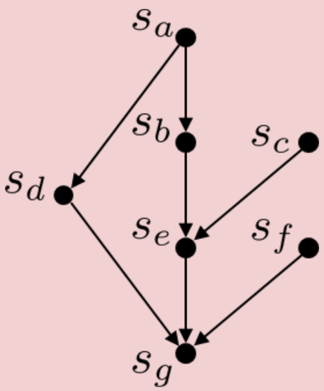
Graph DSP



Signals indexed by nodes of a graph

Shumann 2012
Sandryhaila 2013

New Discrete Lattice SP



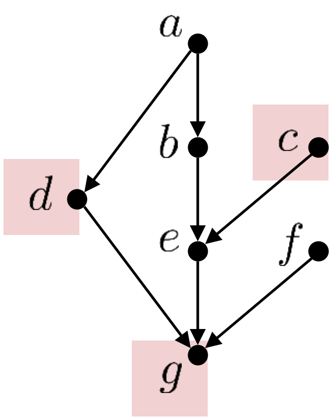
Signals indexed by a meet/join lattice
Instantiation of algebraic signal processing theory

ICASSP 2019

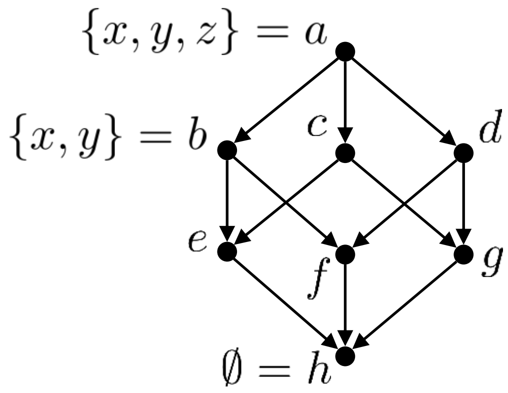
Goal shift, convolution/filtering, Fourier transform, frequency response, sampling, **for lattice signals**

Meet Semilattice

Finite set L with **partial order** $\leq$ and
meet operation $x \wedge y$ (greatest lower bound)



For example, $c \wedge d = g$



Powerset Lattice

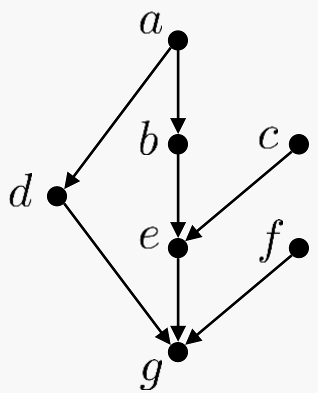
Partial order $\subseteq$, meet $\cap$

Join Semilattice

Analogous

Discrete Lattice Signal Processing

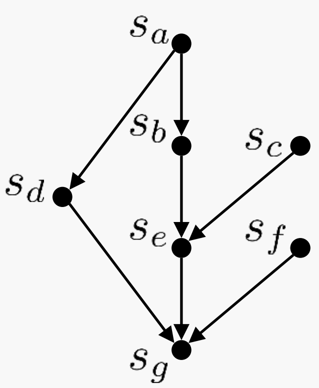
Lattice



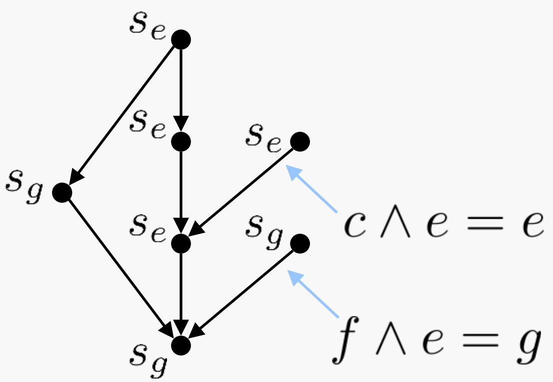
Shift(s) by $q \in L$

$$T_q \mathbf{s} = (s_{x \wedge q})_{x \in L}$$

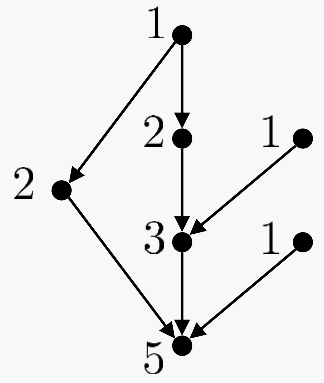
Signal $\mathbf{s} = (s_x)_{x \in L} \in \mathbb{R}^n$



Shifted Signal (by e)

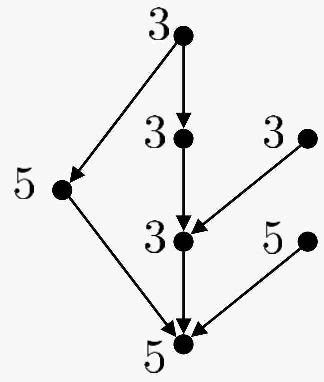


Concrete Example



$T_e \mathbf{s}$

Shifted Example (by e)



Filters: Linear Shift Invariant Systems

Shift(s) by $q \in L$

$$T_q \mathbf{s} = (s_{x \wedge q})_{x \in L}$$

Convolution

$$\mathbf{h} * \mathbf{s} = \left(\sum_{q \in L} h_q s_{x \wedge q} \right)_{x \in L}$$

Filter $\mathbf{h} = (h_q)_{q \in L}$

indexed by lattice

Shift Invariance ✓

Fourier Transform diagonalizes all shifts and filters

$$\widehat{s}_y = \sum_{x \le y} \mu(x, y) s_x$$

indexed by lattice

$$\mu(x, x) = 1, \text{for } x \in L$$

$$\mu(x, y) = -\sum_{x \le z < y} \mu(x, z), \quad \text{for } x \neq y$$

Pure Frequencies eigenvectors of all filters

$$\mathbf{f}^y = (\iota_{y \le x})_{x \in L}, \quad y \in L$$

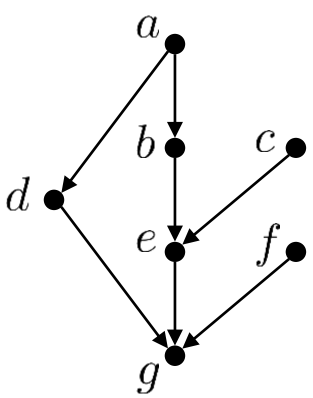
characteristic function

Frequency response eigenvalues

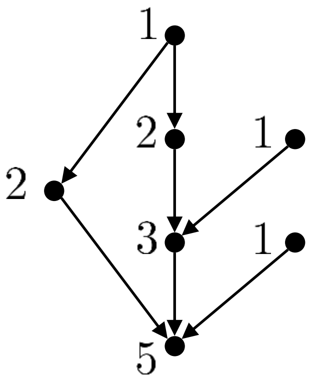
$$\overline{h}_y = \sum_{q \in L,\ y \le q} h_q$$

Algebraic Lattice Theory

Lattice



Signal



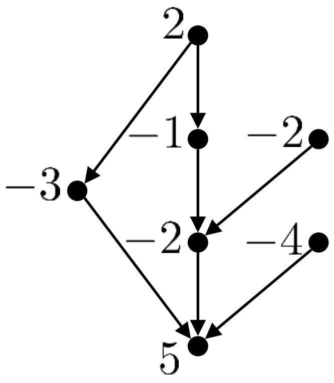
$\mathbf{s}$

Fourier transform

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

F

Spectrum

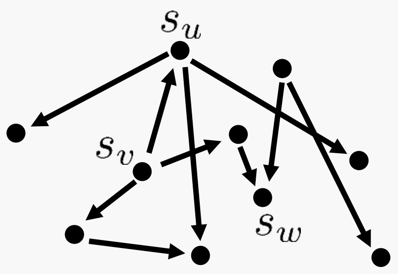


$\widehat{\mathbf{s}} = F\mathbf{s}$

Comparison Graph DSP

Graph DSP

Signals indexed by vertices of a graph



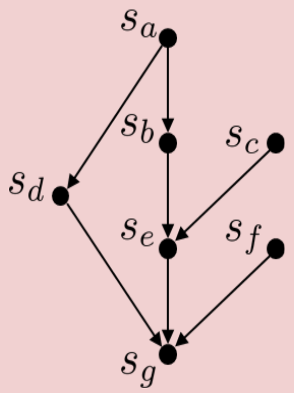
Shift captures adjacency structure

One generating shift
(adjacency or Laplacian)

Shift not always diagonalizable (digraphs)

New Discrete Lattice SP

Signals indexed by a meet/join lattice = special type of graph



Shifts capture partial order structure

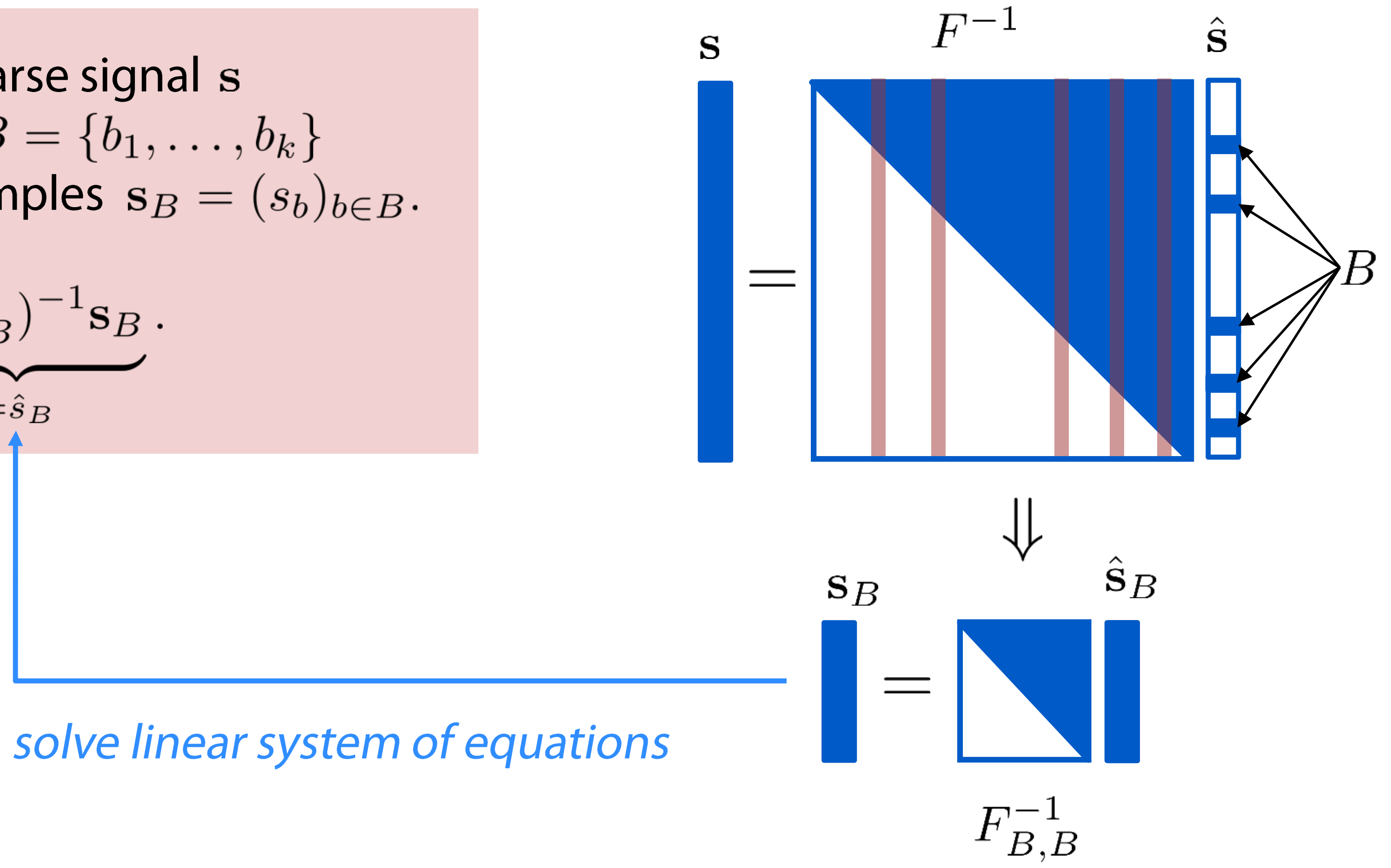
Several generating shifts
(one per 'maximal' element)

Shifts always diagonalizable

Sampling Signals on Meet/Join Lattices

Sampling Theorem: A Fourier sparse signal $\mathbf{s}$ with known support $\mathrm{supp}(\hat{\mathbf{s}}) = B = \{b_1, \ldots, b_k\}$ can be reconstructed from the samples $\mathbf{s}_B = (s_b)_{b \in B}$.

Formally, we have $\mathbf{s} = F^{-1}_{L,B} \underbrace{(F^{-1}_{B,B})^{-1} \mathbf{s}_B}_{=\hat{s}_B}$.



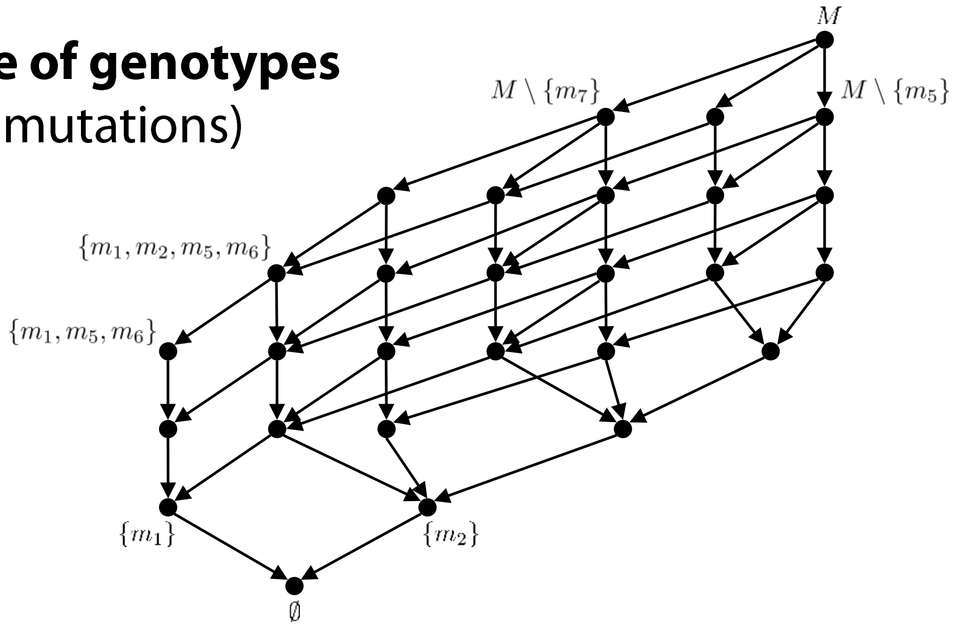
Application 1: Genotype-Phenotype Mappings (HIV RT)

Gene ...GAGAACTTAATAAGAAAACTCAAGA...

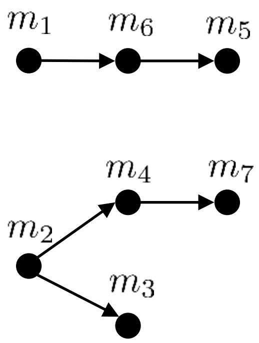
set of mutations

Genotype ...GAGAGCTTAATAAGACAACTCAAGA...

Semilattice of genotypes
(subsets of mutations)



Constraints on mutations



$$M = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$$

Signal

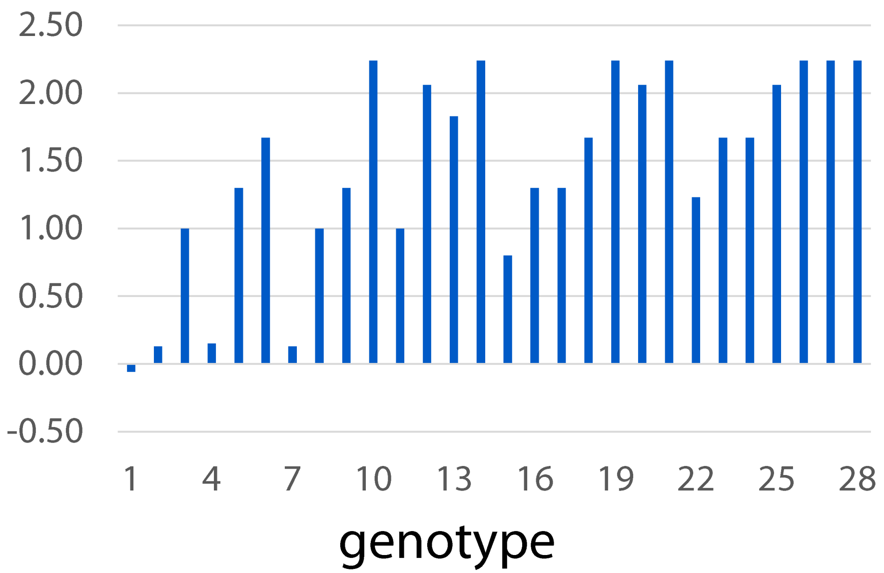
$$s_\emptyset = 0$$
$$s_{\{m_1\}} = 0.15$$
$$s_{\{m_2\}} = 0.13$$
$$s_{\{m_1,m_5,m_6\}} = 1.23$$
$$s_{\{m_1,m_2,m_5,m_6\}} = 1.67$$
$$s_{M\setminus\{m_5\}} = 2.14$$
$$s_{M\setminus\{m_7\}} = 2.14$$
$$s_M = 2.14$$

RT inhibitor resistance

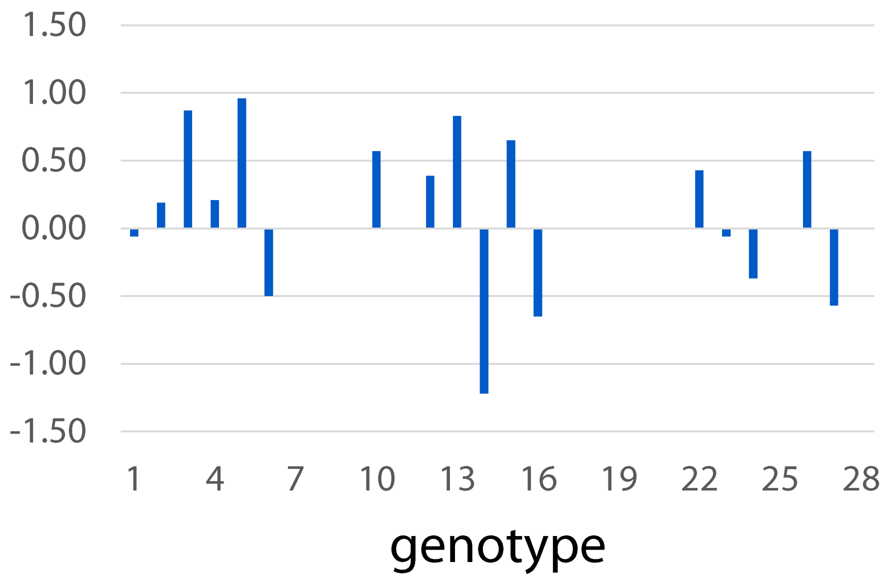
Beerenwinkel, Knupfer and Tresch (2011)

Application 1: Genotype-Phenotype Mappings (HIV RT)

RT inhibitor resistance s



Fourier transform/spectrum $\hat{s}$

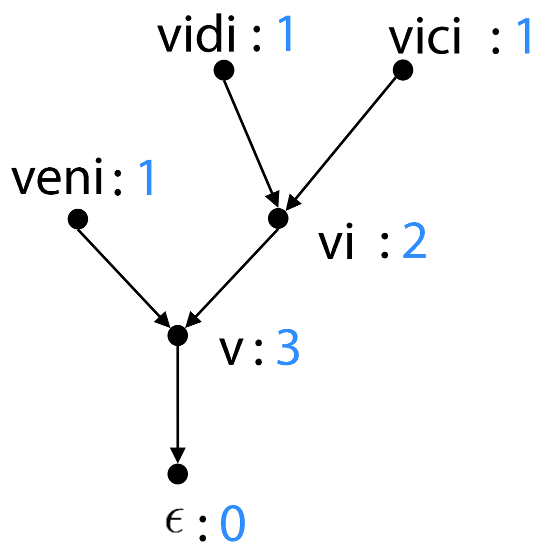


Beerenwinkel, Knupfer and Tresch (2011)

Application 2: Document Representation

Lattice = prefix lattice of words

Signal = prefix count signal

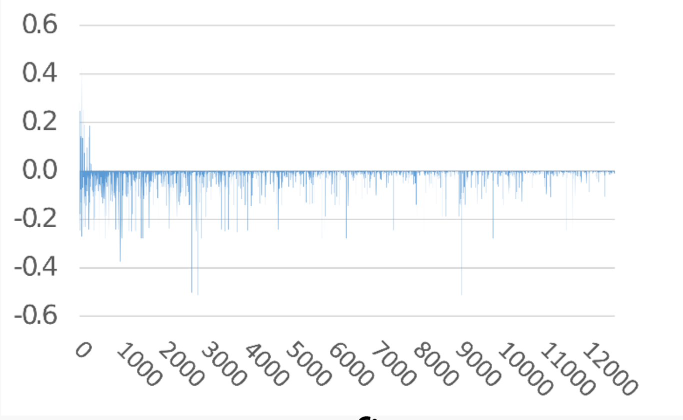


For example: *'veni vidi vici'*

Kritik der reinen Vernunft (Kant 1781)

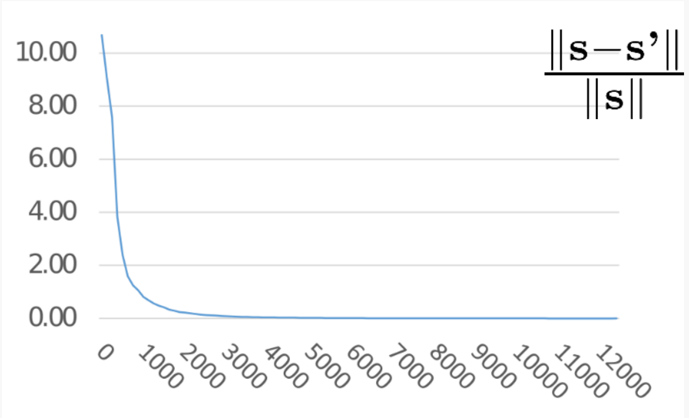
Book ↓	9,952 unique words
Signal = prefix count	12,636 prefixes (lattice)

Prefix count spectrum $\hat{s}$



prefix

Relative reconstruction error



#prefixes used

Application 3: Combinatorial Auctions

Spectrum Auction

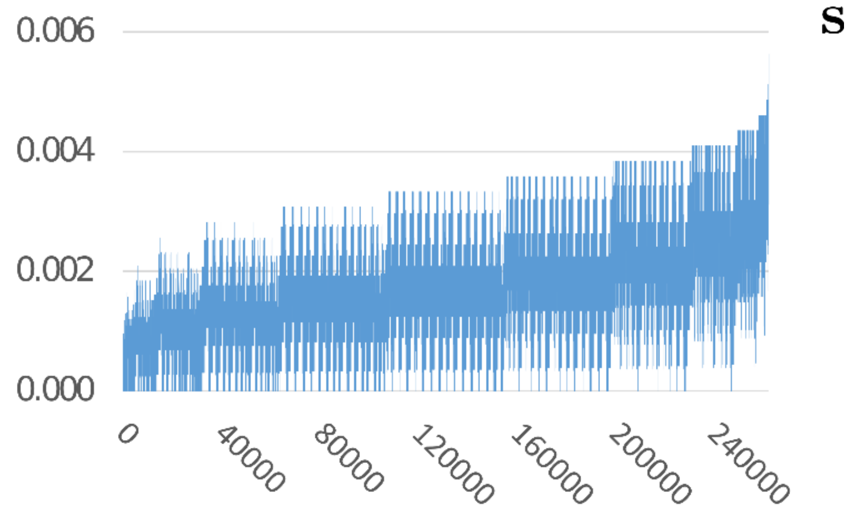
goods = bands of electromagnetic spectrum
bidders = valuation functions $v_i : 2^M \to \mathbb{R}^+$

lattice = powerset, signal = valuation function

GSVM auction 18 goods → 2^{18} valuations/bidder

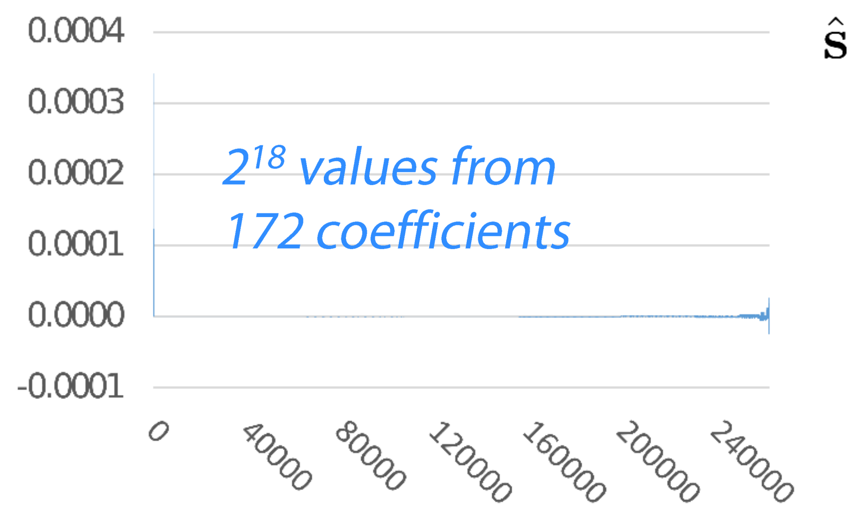
Goeree and Holt (2010)

Bidder valuation function



subset of goods

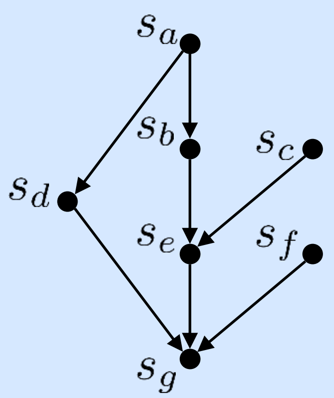
Fourier transform/spectrum



subset of goods

possible application: preference elicitation

Lattice DSP



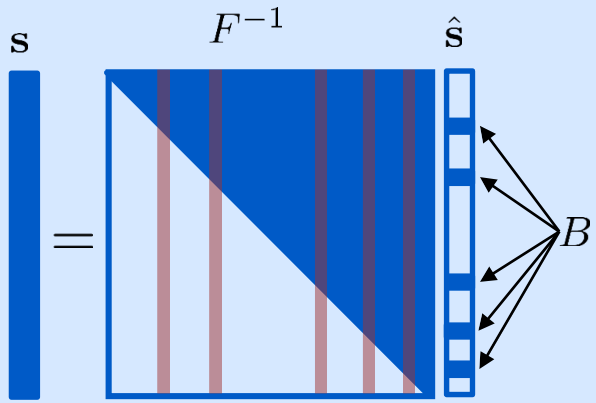
Convolution

$$\mathbf{h} * \mathbf{s} = \left(\sum_{q \in L} h_q s_{x \wedge q}\right)_{x \in L}$$

Fourier Transform

$$\widehat{s}_y = \sum_{x \le y} \mu(x, y) s_x$$

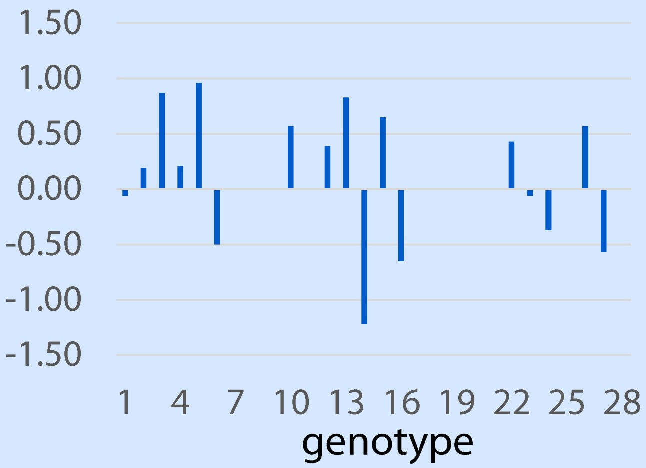
Sampling



$$\mathbf{s} = F^{-1}_{L,B}(F^{-1}_{B,B})^{-1}\mathbf{s}_B$$

reconstruct signal from |B| samples

Possible Applications



1. Genotype-phenotype maps
2. Document representation
3. Set functions (e.g., valuations)
4. ***Powerset Convolutional Neural Network (NeurIPS 2019)***

https://acl.inf.ethz.ch/research/ASP/