Theoretical Guarantees For Poisson Disk Sampling Using Pair Correlation Function

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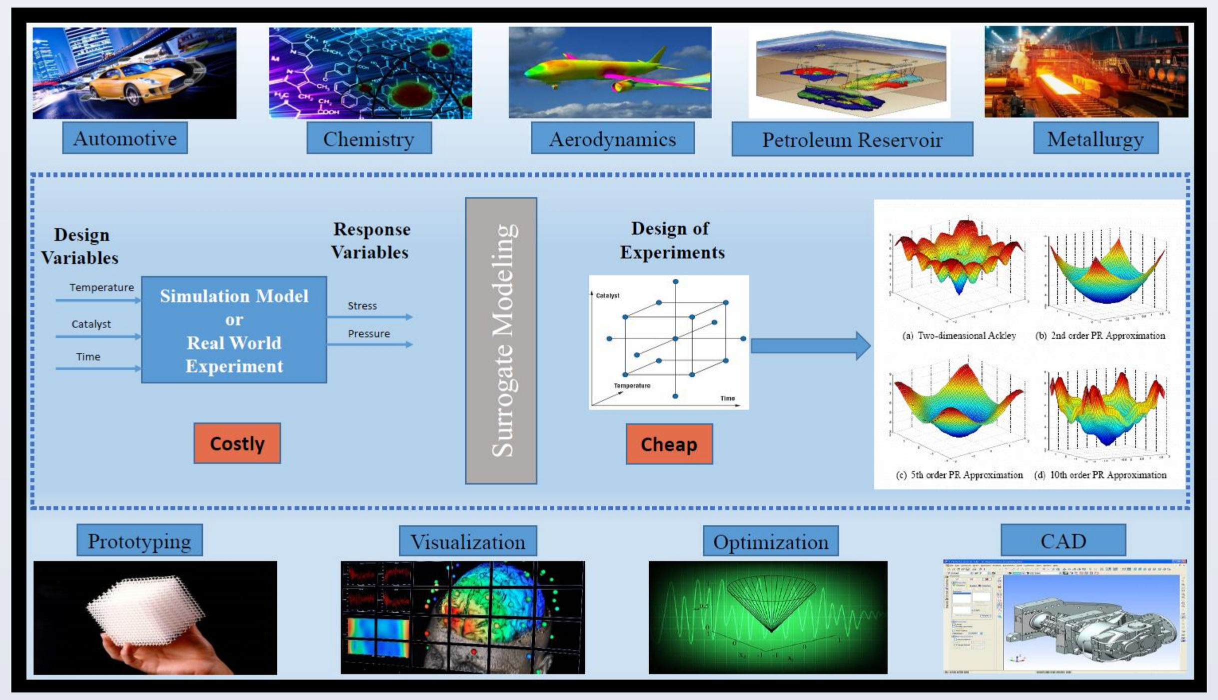
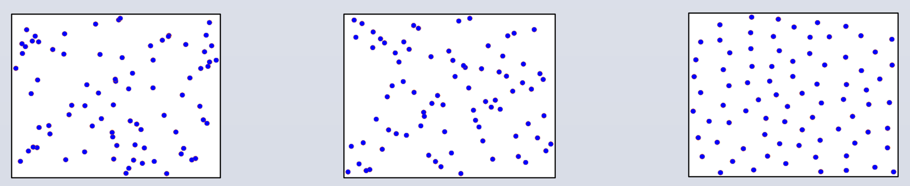
Motivation

In spite of advances in computer capacity and speed, the enormous computational cost of scientific and engineering simulations and time constraints makes it impractical to rely exclusively on simulation codes. A preferable strategy is to replace the expensive simulations using surrogate models

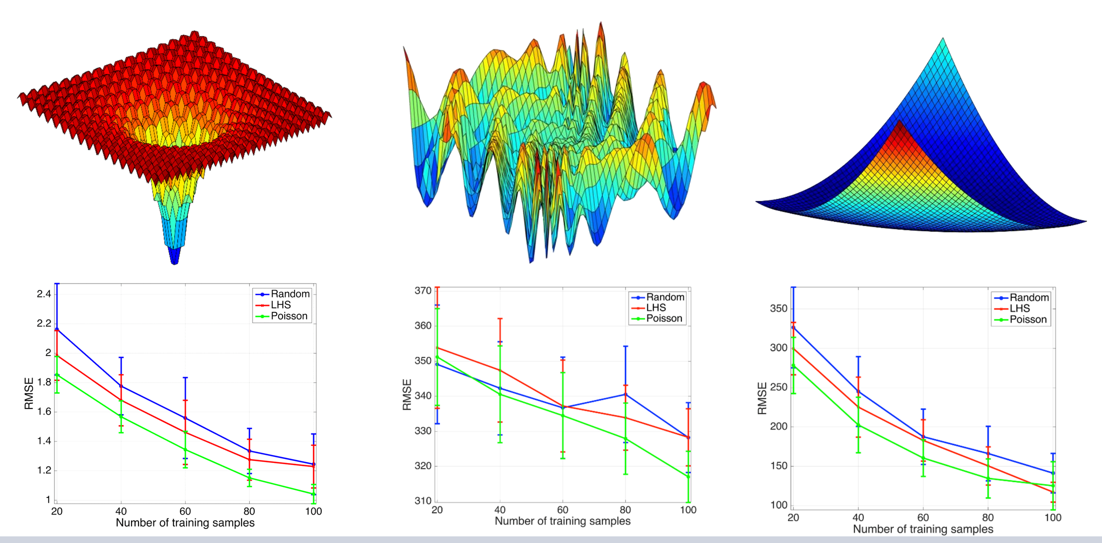


Importance of Experiment Design

- **Surrogate Modeling**: Approximate a high dimensional function defined on *d*-dimensional domain *D*
- Modeling and prediction using Random Forest regressor
- 25 realizations of {20,40,60,80,100} training samples
- Performance is measured based on 2500 testing samples



Random — Latin Hyper Cube — Poisson Disk Sampling



Poisson Disk Sampling (PDS)

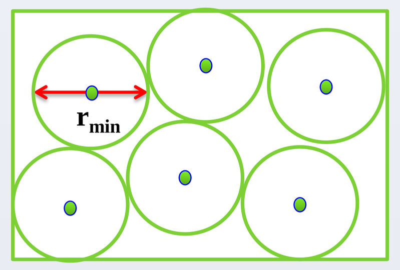
Definition: A set of N point samples $\mathbf{x_i}$ in a sampling domain D are Poisson disk samples, if $X = \{\mathbf{x_i} \in D;\ i = 1, \cdots N\}$ satisfy the following two conditions:

- $\forall \mathbf{x_i} \in X,\ \forall S \subseteq D\ :\ P(\mathbf{x_i} \in S) = \int_S \mathbf{dx}$ → Uniform
- $\forall \mathbf{x_i}, \mathbf{x_j} \in X\ :\ ||\mathbf{x_i} - \mathbf{x_j}|| \geq r_{\min}$ → Min Distance

where $r_{\min}$ is the Poisson disk radius.

Maximal PDS: A PDS is maximal if no more points can be inserted

$$\forall \mathbf{x} \in D,\ \exists \mathbf{x_i} \in X\ :\ ||\mathbf{x} - \mathbf{x_i}|| < r_{\min}.$$



Existing approaches for MPDS experimentally obtain the bounds for N

Challenge: There do not exist any methods to quantitatively understand Poisson disk sampling

PDS using Pair Correlation Function (PCF)

In statistical mechanics, PCF describes how density varies as a function of distance from a reference. PCF, G(r), is related to power spectral density (PSD) via the Hankel Transform

$$P(k) = 1 + \rho(2\pi)^{\frac{d}{2}} k^{1-\frac{d}{2}} H_{\frac{d}{2}-1}\left(r^{\frac{d}{2}-1}(G(r) - 1)\right)$$

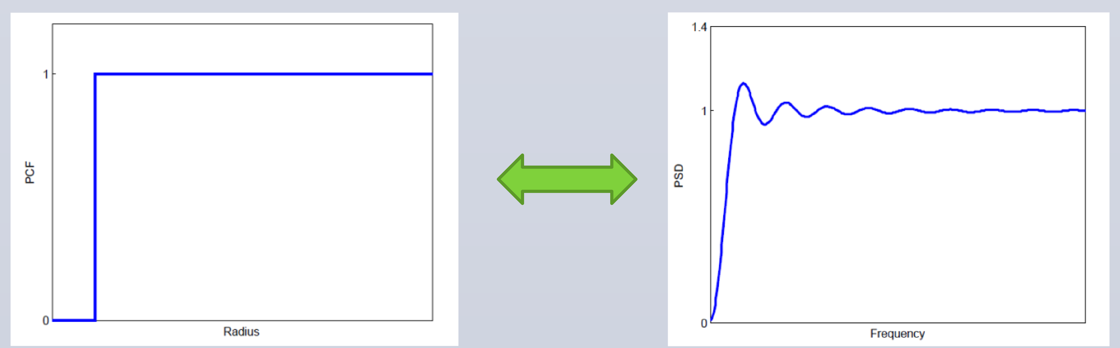
Definition: Given the desired radius $r_{\min}$, Poisson disk sampling is defined in the PCF domain as

$$G(r - r_{\min}) = \begin{cases} 0 & \text{if } r < r_{\min} \\ 1 & \text{if } r \geq r_{\min}. \end{cases}$$

Using the PCF-PSD relation, we can derive the radially averaged PSD of Poisson disk sampling:

$$P(k) = 1 - \rho\left(\frac{2\pi r_{\min}}{k}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(k r_{\min})$$

PAIR CORRELATION — RADIALLY AVERAGED PSD



Theoretical Bounds for PDS

Maximum sample size for a fixed disk radius:

$$\begin{aligned} \text{maximize} \quad & N \\ \text{subject to} \quad & P(k) \geq 0,\ \forall k \\ & G(r - r_{\min}) \geq 0,\ \forall r \end{aligned}$$

Necessary Conditions

$$\text{where } P(k) = 1 - \rho\left(\frac{2\pi r_{\min}}{k}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(k r_{\min}).$$

PSD of PDS

Proposition 1 *For a fixed Poisson disk radius $r_{\min}$, the maximum number of point samples needed for maximal Poisson disk sampling in the sampling region with volume V is given by*

$$N = \frac{V\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d}{2}} r_{min}^{d}}.$$

Maximum radius for a fixed sample size:

Proposition 2 *For a fixed sampling budget N, the maximum Poisson disk radius r_{min} for Poisson disk sampling in the sampling region with volume V is given by*

$$r_{min} = \sqrt[d]{\frac{V\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d}{2}} N}}.$$

Validation of Poisson Disk Sample Designs

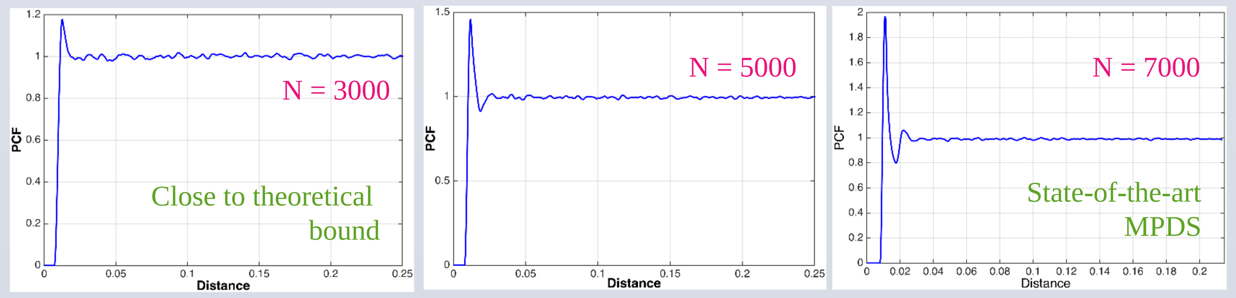


Table 1. The sample sizes and run time (in seconds) required to generate PDS in comparison to the approximate PDS [3]

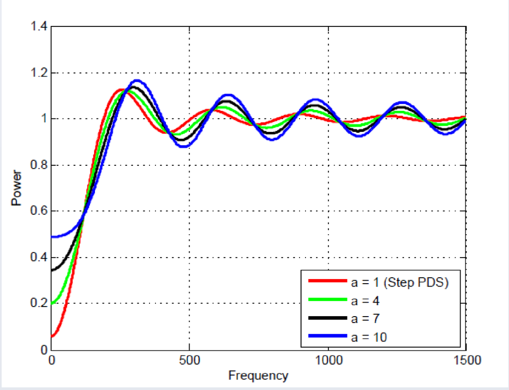
Dimension	Radius	Algorithm in [3]		Proposed PDS	
		N	Time (s)	N*	Time (s)
2	0.005	28098	14.34	12739	10.01
2	0.007	14361	7.34	6499	5.14
2	0.01	7054	3.51	3185	2.49
3	0.03	28776	182.7	8853	88.6
3	0.04	12384	17.55	3735	6.83
4	0.1	10779	1154.59	2028	305.3
4	0.2	849	18.22	127	3.25

Empirical Analysis of Sampling Quality

Desired Properties:

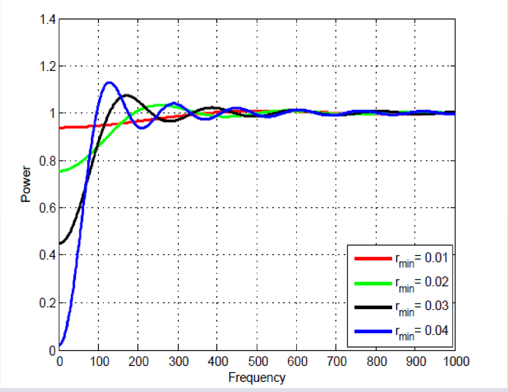
- Zero for low frequencies which indicates the range of frequencies that can be represented with almost no aliasing
- Constant for high frequencies which reduces the risk of aliasing

Importance of uniformity to control aliasing



Increased peak height results in higher low frequency aliasing and larger high frequency oscillations

Impact of the choice of disk radius



Trade-off :

- Spectrum tends to be close to zero at low frequencies
- Significant increase in oscillations at high frequencies

Conclusions

- Pair Correlation analysis allows theoretical analysis of Poisson disk sampling
- Upper bounds on sample size enable early termination of existing Maximal PDS algorithms
- Effect of the choice of disk radius on the resulting spectrum reveals trade-off between low frequency aliasing and high frequency oscillations

References

1. A. Lagae and P. Dutre, “A comparison of methods for generating poisson disk distributions,” Computer Graphics Forum, vol. 27, no. 1, pp. 114–129, 2008.
2. D. Heck, T. Schlomer, and O. Deussen, “Blue noise sampling with controlled aliasing,” ACM Trans. Graph., vol. 32, no. 3, pp. 25:1–25:12, Jul. 2013.
3. M. S. Ebeida, S. A. Mitchell, A. Patney, A. A. Davidson, and J. D. Owens, “A simple algorithm for maximal poisson disk sampling in high dimensions,” Computer Graphics Forum, vol. 31, no. 2pt4, pp. 785–794, 2012.