BAYESIAN TUNING FOR SUPPORT DETECTION AND SPARSE SIGNAL ESTIMATION VIA ITERATIVE SHRINKAGE-THRESHOLDING

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CRISP
Towards Compressive Information Processing Systems
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1 -- A Bayesian view of sparse signal recovery via Lasso

Problem: sparse signal recovery from compressed data

- unknown sparse signal: $x^\star \in \mathbb{R}^n$ with $|\{i : x_i^\star \neq 0\}| \leq k \ll n$
- set of observations:

$$y = Ax^\star + \eta$$

- sensing matrix: $A \in \mathbb{R}^{m\times n}$ with $m \ll n$
- white gaussian noise: $\eta \in \mathbb{R}^m$
- the signal prior is not known

Lasso solution:

$$\min \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1, \ \lambda > 0$$

- equivalent to a ML estimation, assuming $x_i^\star$ i.i.d. Lap$(0, \lambda^{-1})$:
- is invariant to the signal prior

Aim: new methods for learning sparsity models able to
- ▷ reduce the number of measurements required for reconstruction
- ▷ improve the computational complexity in the estimation

2 -- This work: weighted Lasso via 2LMM

Our solution: new parametric model for the signal able to:
- ▷ **combine support detection and estimation**
- ▷ **preserve the simplicity coming from Laplace assumption**

2LMM: x is a random variable with i.i.d. entries

$$x_i = z_i u_i + (1 - z_i)v_i,$$
$$u_i \sim \text{Lap}(0, \alpha), v_i \sim \text{Lap}(0, \beta), \quad 0 \approx \alpha < \beta$$
$$z_i \sim \text{Ber}(1 - p) \quad p \ll 1/2$$

- ML from complete data: weighted Lasso solution

$$\min \frac{1}{2}\|y - Ax\|_2 + \sum_{i=1}^{n}\left[\frac{\pi_i|x_i| + \epsilon/n}{\alpha} + \frac{(1-\pi_i)|x_i| + \epsilon/n}{\beta} - \pi_i \log\frac{1-p}{\alpha} - (1-\pi_i)\log\frac{p}{\beta}\right]$$

z_i hidden data, $\pi_i = \mathbb{P}(z_i = 1|x, \alpha, \beta, p)$, x_i visible data, α, β, p mixture parameters

3 -- 2LMM tuned iterative shrinkage-thresholding

Aim: Improve existing methods for Lasso minimization based on shrinkage-thresholding via 2-LMM tuning

ISTA (Daubechies & al., 2004) Start from $x(0) = 0$ and compute

$$x^{(t+1)} = \eta_{\lambda\tau}[x^{(t)} + \tau A^{\mathsf{T}}(y - Ax^{(t)})].$$

where $\tau \in (0, 2\|A\|_2^{-2})$, η_γ thresholding function

$$\eta_\gamma[x] = \text{sgn}(x)\max(|x| - \gamma, 0)$$

+ Convergence: analytical conditions for convergence to a minimum
− Rate: speed is sensitive to local well-conditioning of the matrix A

Other alternatives: better rate of convergence and phase transitions, i.e. FISTA (Beck&Teboulle, 2010), AMP (Donoho & al., 2009).

Proposed solution: 2LMM-tuning

- ▶ incorporate support detection at each iteration
- ▶ iterative minimization of weighted Lasso
- ▶ use Expectation Maximization to learn the mixture parameters

2LMM-ISTA

Initialization: set K, $p = K/n$ $\alpha^{(0)} = \alpha_0$, $\pi^{(0)} = \mathbb{1}$, $\epsilon^{(0)} = 1$;
for $t = 1, \ldots, StopIter$
- ℓ_1-weights: $\omega_i^{(t+1)} = \pi_i^{(t)}/\alpha^{(t)} + (1 - \pi_i^{(t)})/\beta^{(t)}$
- Gradient/Thresholding step:

$$x^{(t+1)} = \eta^{\mathsf{S}}_{\lambda\tau\omega^{(t+1)}}(x^{(t)} + \tau A^\top(y - Ax^{(t)}))$$

- Posterior distribution evaluation:

$$\pi^{(t+1)} = \sigma_{n-K}\left[\mathbb{P}(z_i = 1|x^{(t+1)}, \alpha^{(t)}, \beta^{(t)}, p)\right]$$

- Regularization parameter:

$$\epsilon^{(t+1)} = \min\left(\frac{1}{\log(t+1)} + c\|x^{(t+1)} - x^{(t)}\| + \frac{r(x^{(t+1)})_{K+1}}{n}, \epsilon^{(t)}\right)$$

- Parameters estimation:

$$\alpha^{(t+1)} = \frac{\sum_i \pi_i^{(t+1)}|x_i^{(t+1)}| + \epsilon^{(t+1)}}{\|\pi^{(t+1)}\|_1} \qquad \beta^{(t+1)} = \frac{\sum_i (1-\pi_i^{(t+1)})|x_i^{(t+1)}| + \epsilon^{(t+1)}}{\|\mathbb{1} - \pi^{(t+1)}\|_1}$$

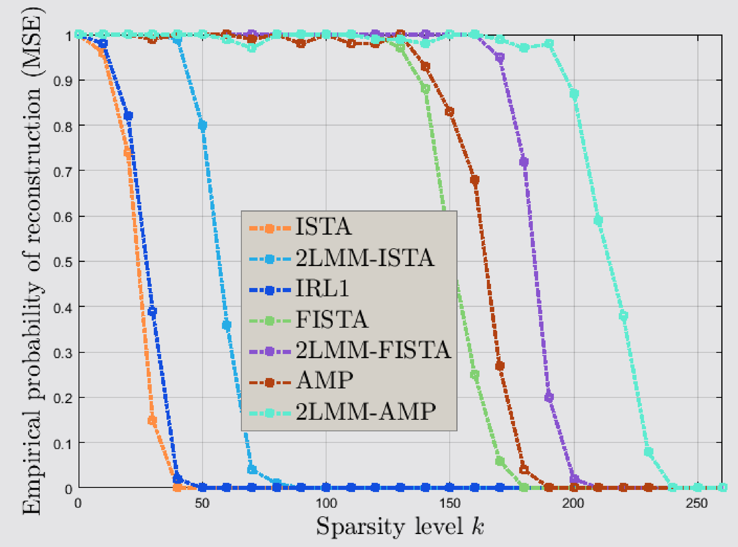
end for

+ Convergence: analytical conditions for convergence to a limit point
+ Rate: faster than classical methods

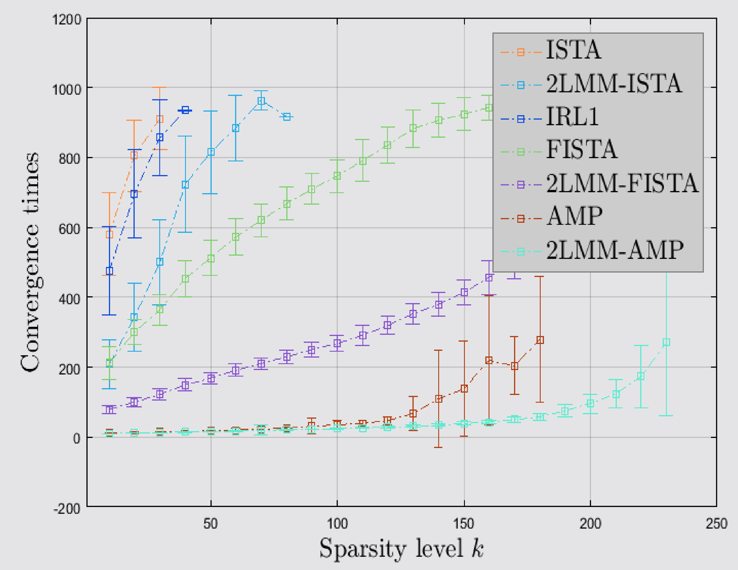
2LMM-tuning can also be applied to other shrinkage-thresholding methods, e.g. 2LMM-FISTA, 2LMM-AMP.

4 -- Numerical experiments

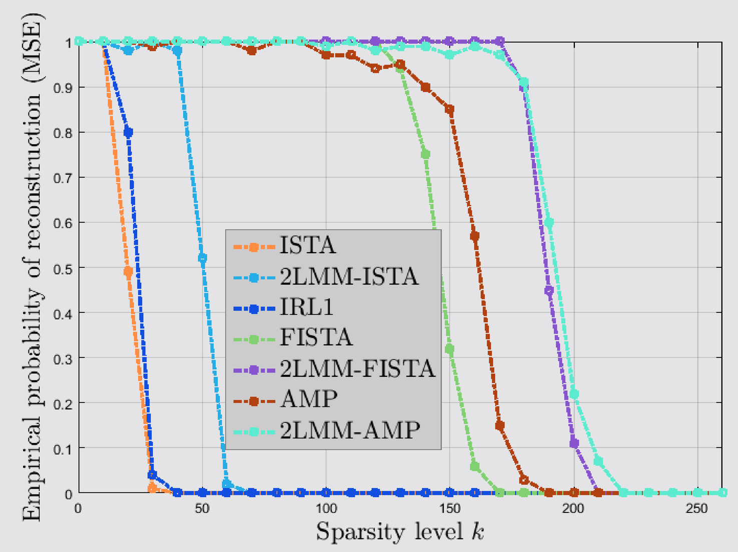
Reconstruction from noise-free measurements:
- ▶ Sparse signals of length $n = 512$, number of measurements $m = 350$
- ▶ Gaussian sensing matrices: $A_{ij} \sim N(0, 1/m)$



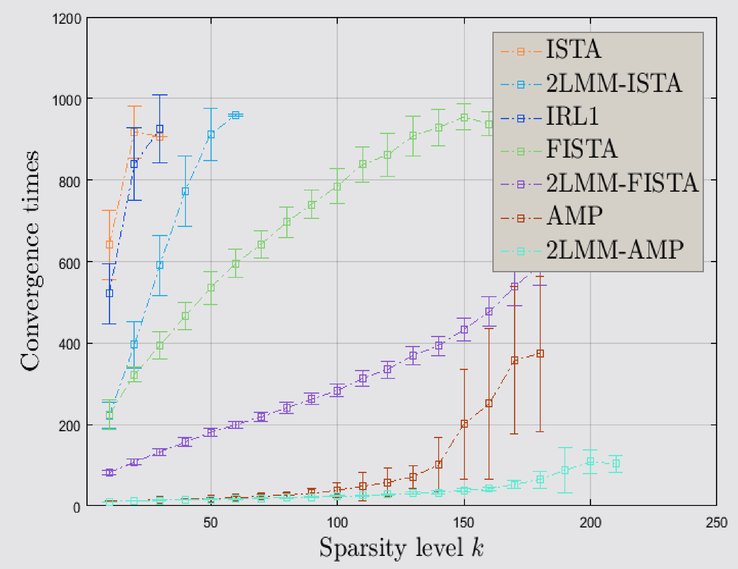
Sparse Bernoulli-Gaussian signals: success (MSE $< 10^{-4}$)



Sparse Bernoulli-Gaussian signals: average running times (CPU Times in [s])

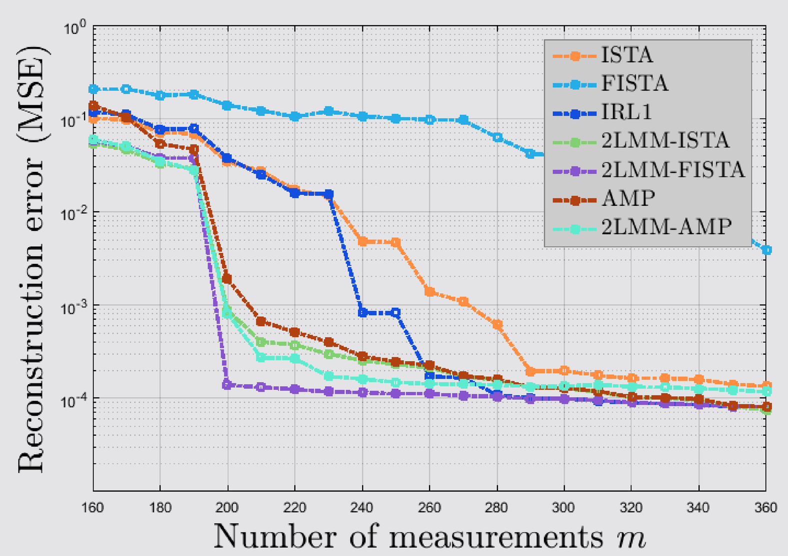


Sparse Bernoulli-Binomial signals: success (MSE $< 10^{-4}$)



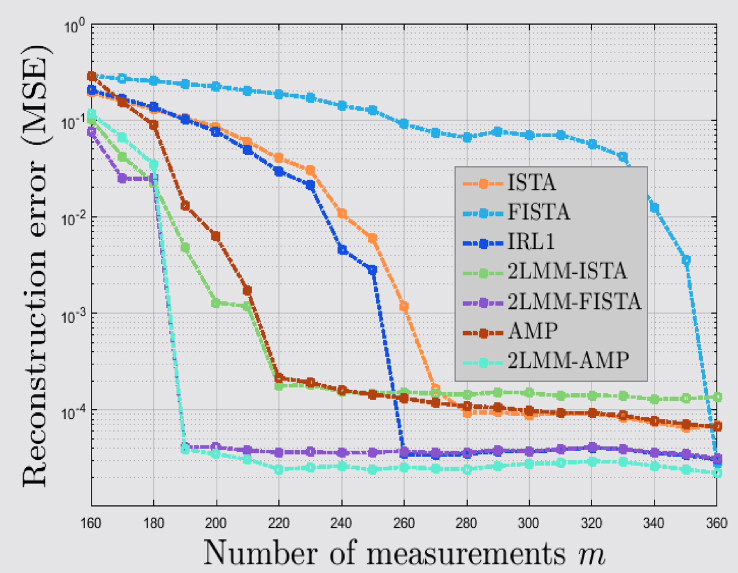
Sparse Bernoulli-Binomial signals: average running times (CPU Times in [s])

Reconstruction in imperfect scenarios:



Not exactly sparse signals: $x^\star = x_0 + \xi$

- x_0 exactly 56-sparse Bernoulli-Gaussian signal of length $n = 512$;
- ξ random noise with $\sigma = 0.01$



Reconstruction from noisy measurements

- $x^\star$ exactly 56-sparse Bernoulli-Gaussian signal of length $n = 512$;
- η random noise with $\sigma = 0.01$

Additional material: C. Ravazzi and E. Magli, "Laplace mixture models for efficient and accelerated compressed sensing", submitted to IEEE Trans. on Signal Processing.

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