# A Stochastic Model of Block Segmentation Based on the Quadtree and the Bayes Code for It

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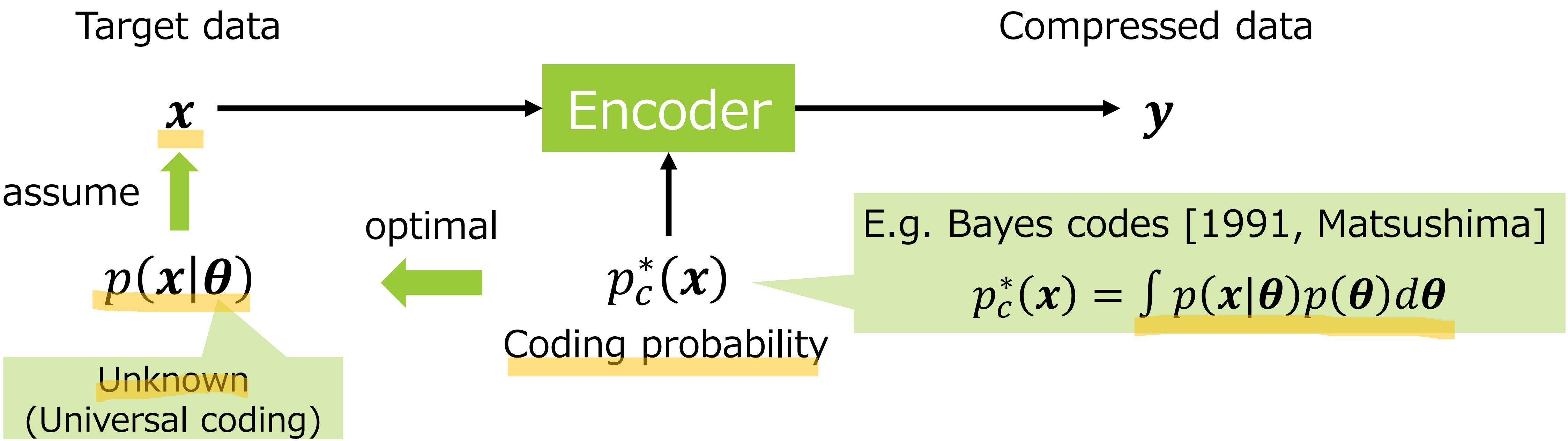
# Lossless compression of general data

## ■Information theoretical approach

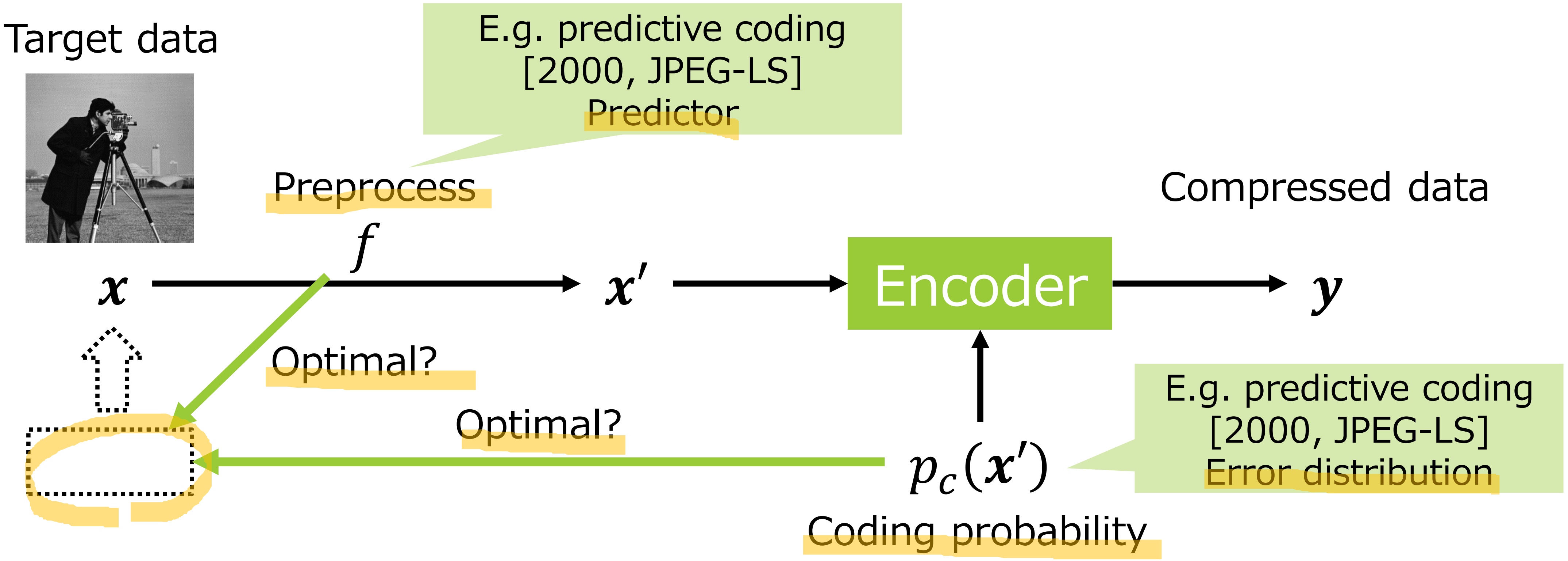



## ■The important factors

- The efficiency of the algorithm to calculate $p_c^*(\boldsymbol{x})$
- The flexibility of the stochastic model $p(\boldsymbol{x}|\boldsymbol{\theta})$

# Lossless image compression

- Approaches in most previous studies



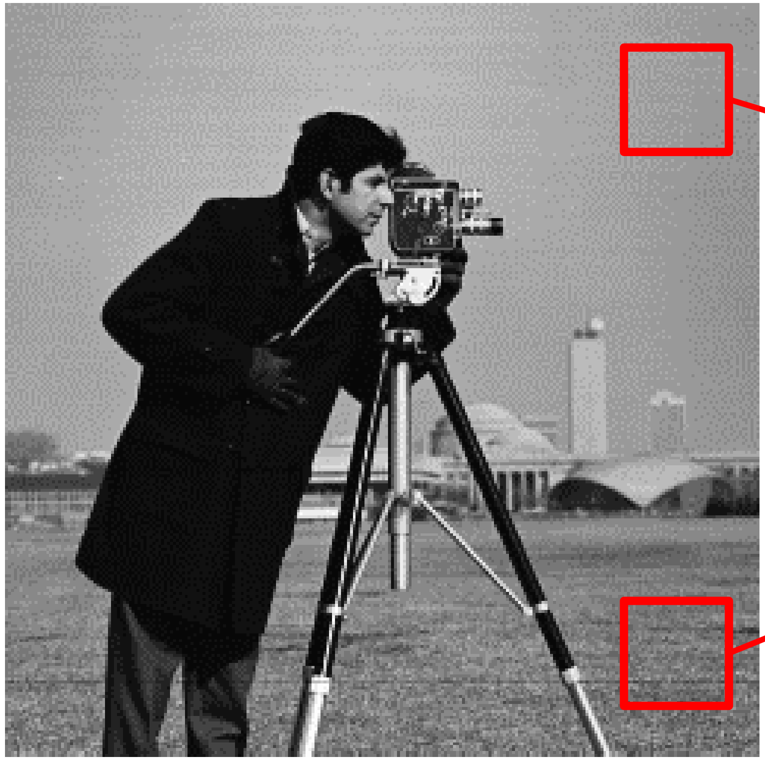

- Difficulty to discuss the optimality of $f$ and $p_c(\boldsymbol{x}')$

# Purposes of this study

1. We propose a stochastic model that effectively represents target images.
2. We derive an optimal coding probability.
3. We derive an efficient algorithm to calculate it without loss of the optimality.

# Target data

- Images with non-stationarity



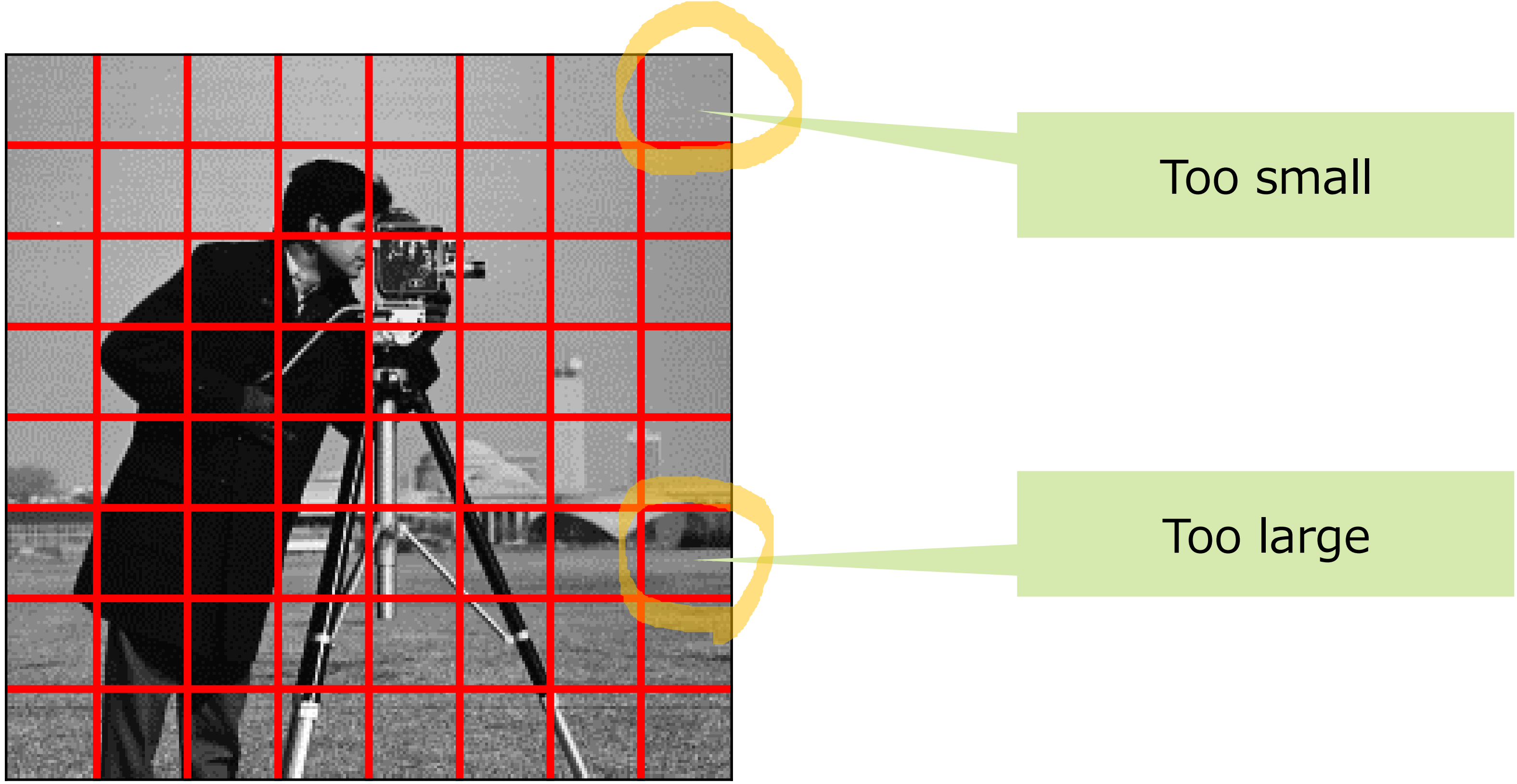
$$p(\boldsymbol{x}|\boldsymbol{\theta})$$

$$\neq$$

$$p(\boldsymbol{x}|\boldsymbol{\theta}')$$

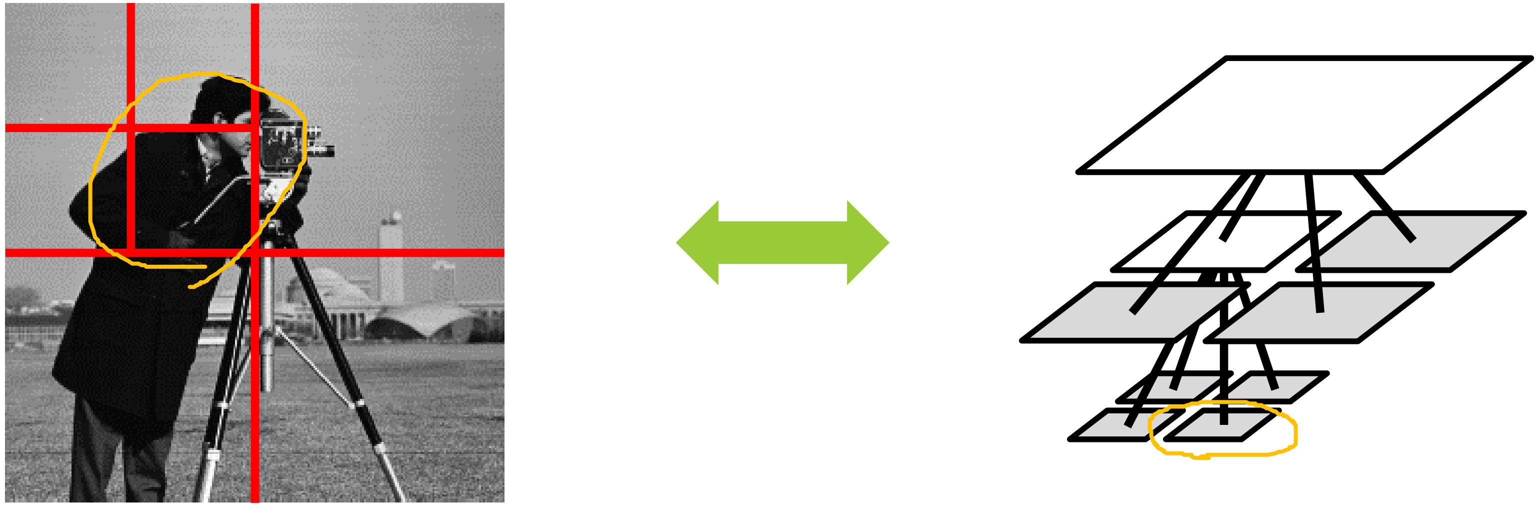
# Trivial way

- Divide the image into fixed size blocks
- Assume different stochastic models to them



# Quadtree

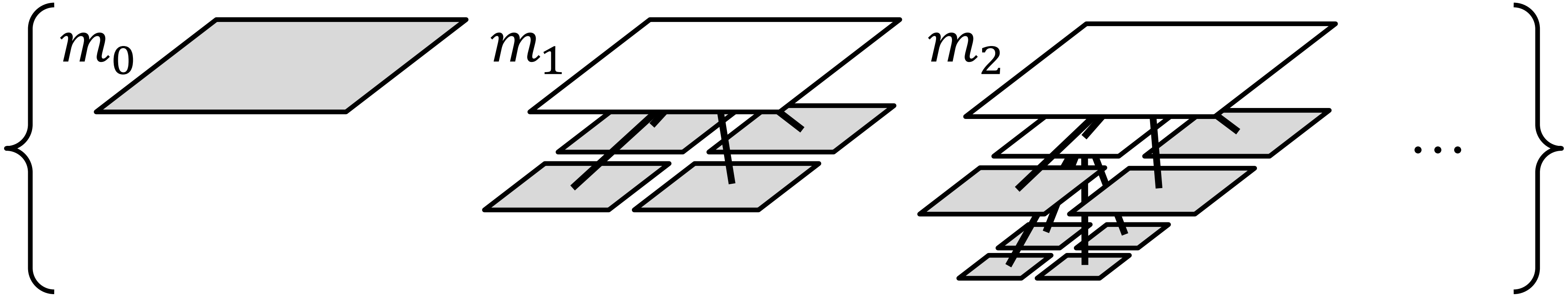
- It effectively represents variable block size segmentation
- 



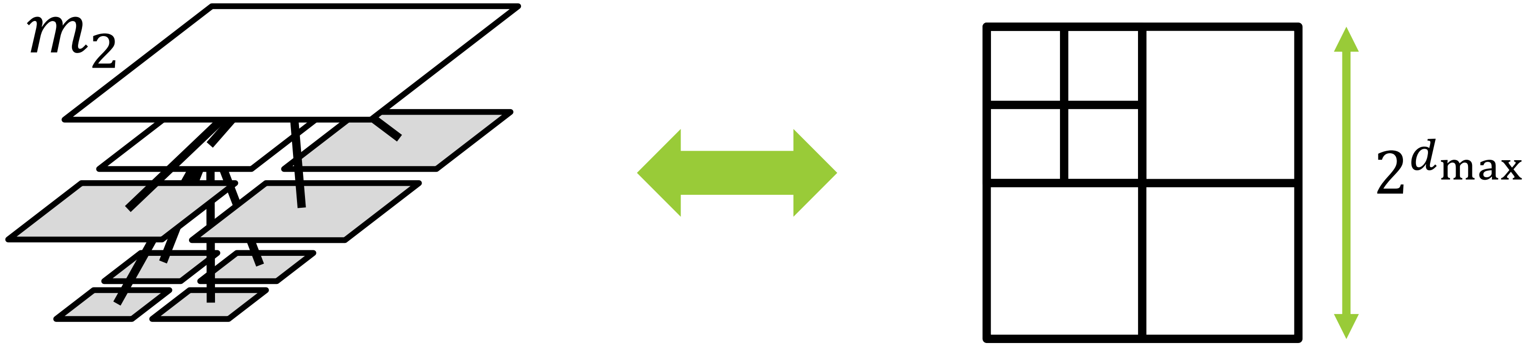
- Previous studies regard it just as a procedure
- In this study, we regard the quadtree as a part of stochastic generative model of the images.

# Proposed stochastic model

- Let both of the width and height are $2^{d_{\max}}$.
- Consider the set of the quadtrees whose depth $\leq d_{\max}$.

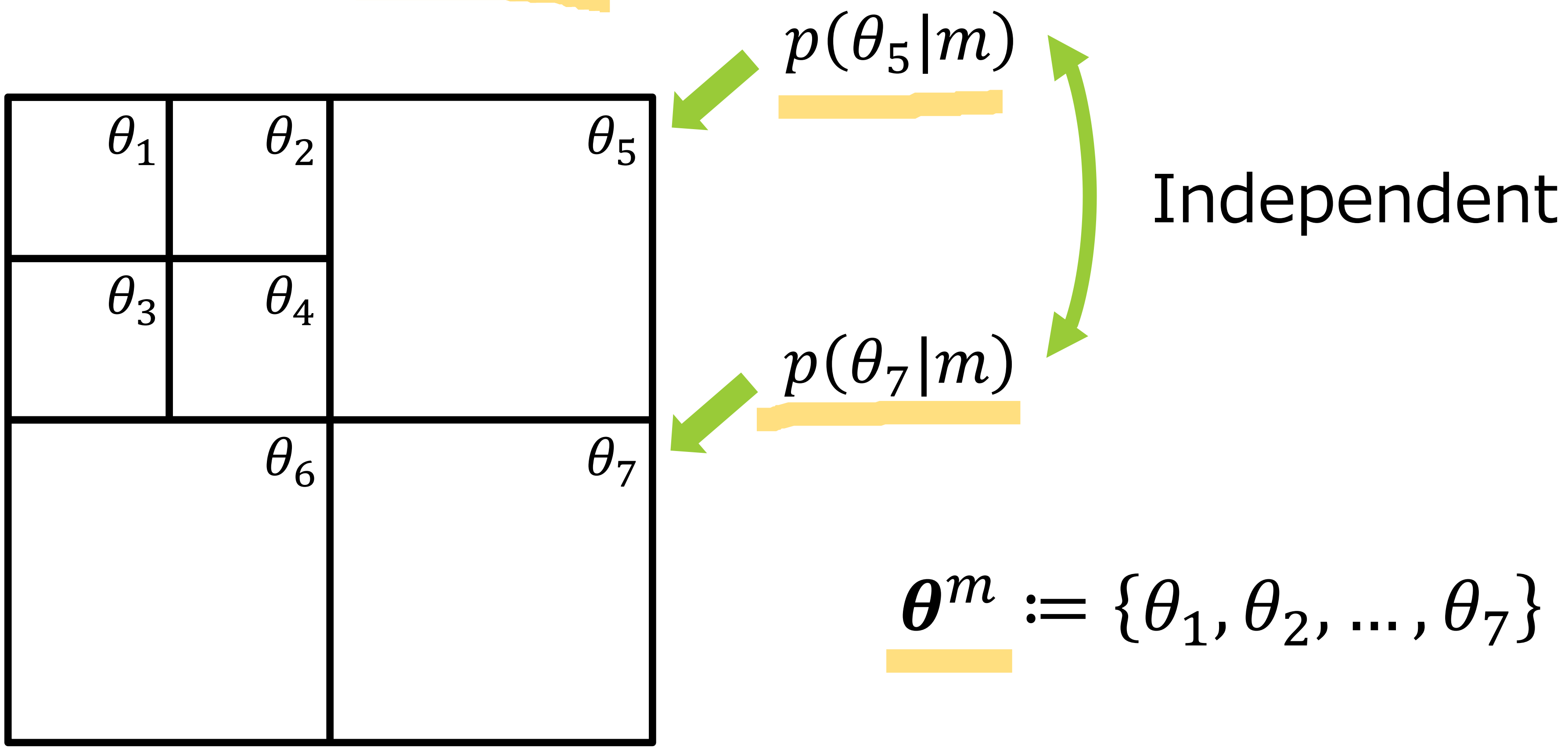
$\mathcal{M} =$ 


- One of them is chosen with probability $p(m)$.



# Proposed stochastic model

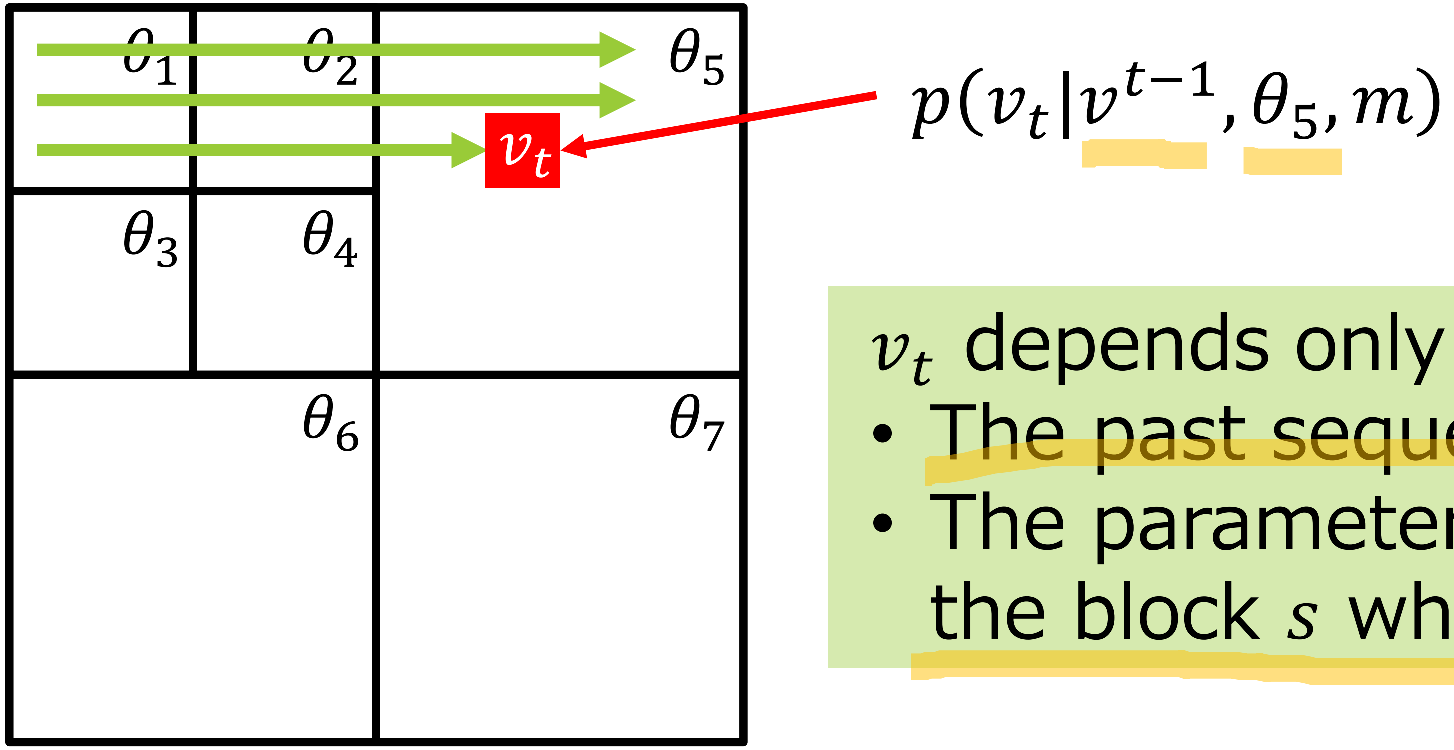
- Parameter $\theta_s$ is independently assigned to each block $s$ with probability $p(\theta_s|m)$.




$$\boldsymbol{\theta}^m := \{\theta_1, \theta_2, \dots, \theta_7\}$$

# Proposed stochastic model

- Pixel value $v_t$ at block $s$ is generated in order of the raster scan with probability $p(v_t|v^{t-1}, \theta_s, m)$.




$v_t$ depends only on
- The past sequence $v^{t-1}$
- The parameter $\theta_s$ of the block $s$ which contains $v_t$

# The Bayes code

- We cannot use $p(v_t|v^{t-1}, \theta^m, m)$ because true $m$ and $\boldsymbol{\theta}^m$ are unknown.
- We estimate it by $\hat{p}_c(v_t|v^{t-1})$ in Bayesian manner.

- Optimal coding probability $p_c^*(v_t|v^{t-1})$ for our model

$$p_c^*(v_t|v^{t-1}) = \sum_{m \in \mathcal{M}} p(m|v^{t-1}) \int p(v_t|v^{t-1}, \boldsymbol{\theta}^m, m) p(\boldsymbol{\theta}^m|v^{t-1}, m) \, \mathrm{d}\boldsymbol{\theta}^m$$

- Three computationally hard parts
  - 1. The summation w.r.t. $m$ ← A recursive structure of quadtree
  - 2. The posterior $p(m|v^{t-1})$ ← Special prior (Detailed in the paper)
  - 3. The integral w.r.t. $\boldsymbol{\theta}^m$ ← Conjugate prior

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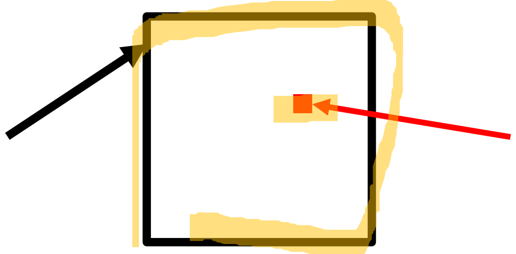
# The derived algorithm

- Consider the following case

The whole image

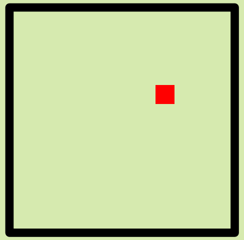


$v_t$: The pixel we want to compress

- $p_c^*(v_t|v^{t-1})$

$$= \underbrace{\left(1 - g_{s_\lambda|t-1}\right)}_{\text{Posterior}} \underbrace{q(v_t|v^{t-1}, s_\lambda)}_{\text{Coding Probability}} + \underbrace{g_{s_\lambda|t-1}}_{\text{Posterior}} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}_{\text{Coding Probability}}$$

The model whose minimal block which contains $v_t$ is root block $s_\lambda$



The other models

# The derived algorithm

- $p_c^*(v_t|v^{t-1})$
$$= (1 - g_{s_\lambda|t-1})q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1}p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})$$
$$= (1 - g_{s_\lambda|t-1})q(v_t|v^{t-1}, s_\lambda) + \underbrace{g_{s_\lambda|t-1}(1 - g_{s_{\lambda\,\mathrm{child}}|t-1})}_{\text{Posterior}}\underbrace{q(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}_{\text{Coding Probability}}$$
$$+ \underbrace{g_{s_\lambda|t-1}g_{s_{\lambda\,\mathrm{child}}|t-1}}_{\text{Posterior}}\underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child\,child}})}_{\text{Coding Probability}}$$

# The derived algorithm

- $p_c^*(v_t|v^{t-1})$

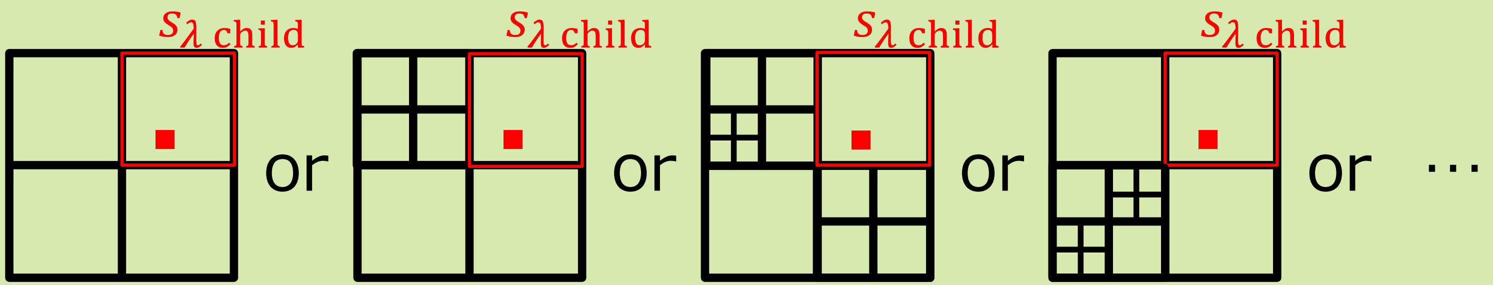
$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})$$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda)$$

$$+ g_{s_\lambda|t-1}\left(1 - g_{s_{\lambda\,\mathrm{child}}|t-1}\right) q(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})$$

Posterior Coding Probability

$$+ \ldots_1 g_{s_{\lambda\,\mathrm{child}}|t-1} p_c(v \ldots s_{\lambda\,\mathrm{child\,child}})$$




The models whose minimal block which contains $v_t$ is $s_{\lambda\,\mathrm{child}}$

# The derived algorithm

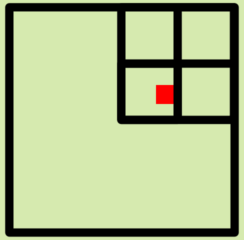
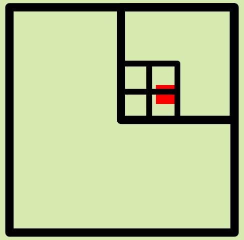
- $p_c^*(v_t|v^{t-1})$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}$$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda)$$

$$+ \underbrace{g_{s_\lambda|t-1}\left(1 - g_{s_{\lambda\,\mathrm{child}}|t-1}\right)}_{\text{Posterior}} \underbrace{q(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}_{\text{Coding Probability}}$$

$$+ \underbrace{g_{s_\lambda|t-1} g_{s_{\lambda\,\mathrm{child}}|t-1}}_{\text{Posterior}} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child\,child}})}_{\text{Coding Probability}}$$

The other models:  or  or ⋯

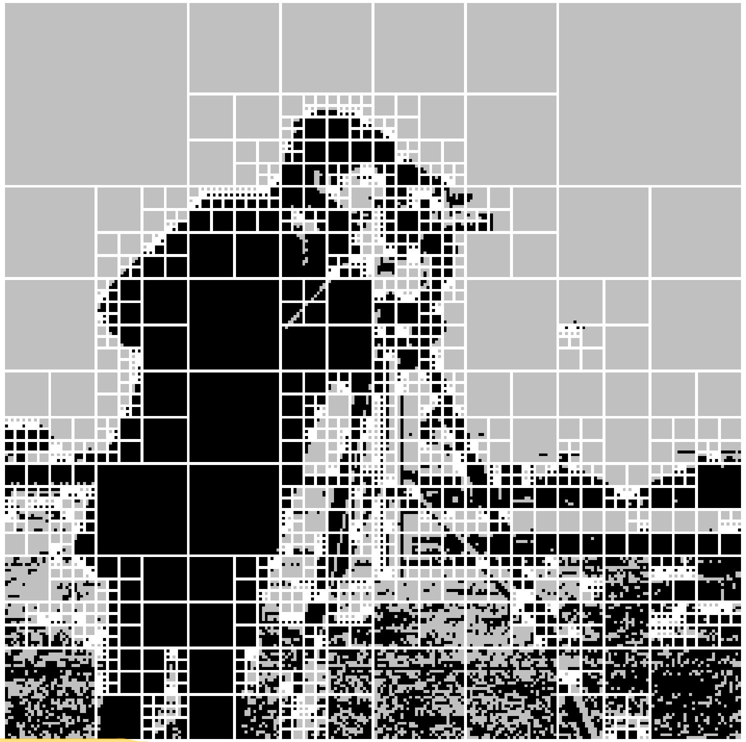
# Experiment 1

- Purpose: To confirmation of the Bayes optimality
- Setting:
  - We generated 1000 binary images (64x64) as follows:
    1. Generate $m$ according to $p(m)$ (detailed in the paper).
    2. Generate $\theta_s$ according to $Beta(\theta_s|\alpha,\beta)$ for each block $s$.
    3. Generate $v_t$ according to $Bern(v_t|\ \theta_s)$ for each block $s$.
  - We compressed them by
    - the proposed method
    - the method with fixed size block segmentation
- Result: Average coding rates (bit/pel)

| Proposed | Fixed size 4 | Fixed size 8 | Fixed size 16 |
|---|---|---|---|
| **0.619** | 0.705 | 0.659 | 0.679 |

# Experiment 2

- Purpose: To demonstrate the model flexibility
- We compressed the binarized image of the cameraman.
- Result: MAP estimated model $m^{\mathrm{MAP}} = \mathrm{argmax}_{m \in \mathcal{M}} p(m|\boldsymbol{v})$.



# Conclusion and future works

- Conclusion
  - We proposed the novel stochastic model with the quadtree.
  - We derived the Bayes optimal coding probability for it.
  - We derived the efficient algorithm to calculate it.

- Future works
  - Our proposed model can be used for other problems
    - Image generation
    - Image inpainting
    - Future extraction