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Entropy Coders Based on the Splitting of Lexicographic Intervals

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Abstract

We propose a technique that performs entropy coding by splitting lexicographic intervals. We mention the main characteristics of our technique, where most of the characteristics definitely apply, by design, and the others are expected to apply, after empirical or theoretical demonstrations are provided. Our technique is (or, at least, should be):

- based on automata quite similar to Mealy machines;
- fast in encoding and decoding;
- able to achieve arbitrarily low redundancy;
- designed to require a small number of states;
- able to decode forwards (making it suitable for streaming);
- able to handle skewed probability distributions;
- intended for stationary memoryless sources;
- a kind of variable-to-fixed coding; and
- able to handle finite source alphabets of arbitrary sizes.

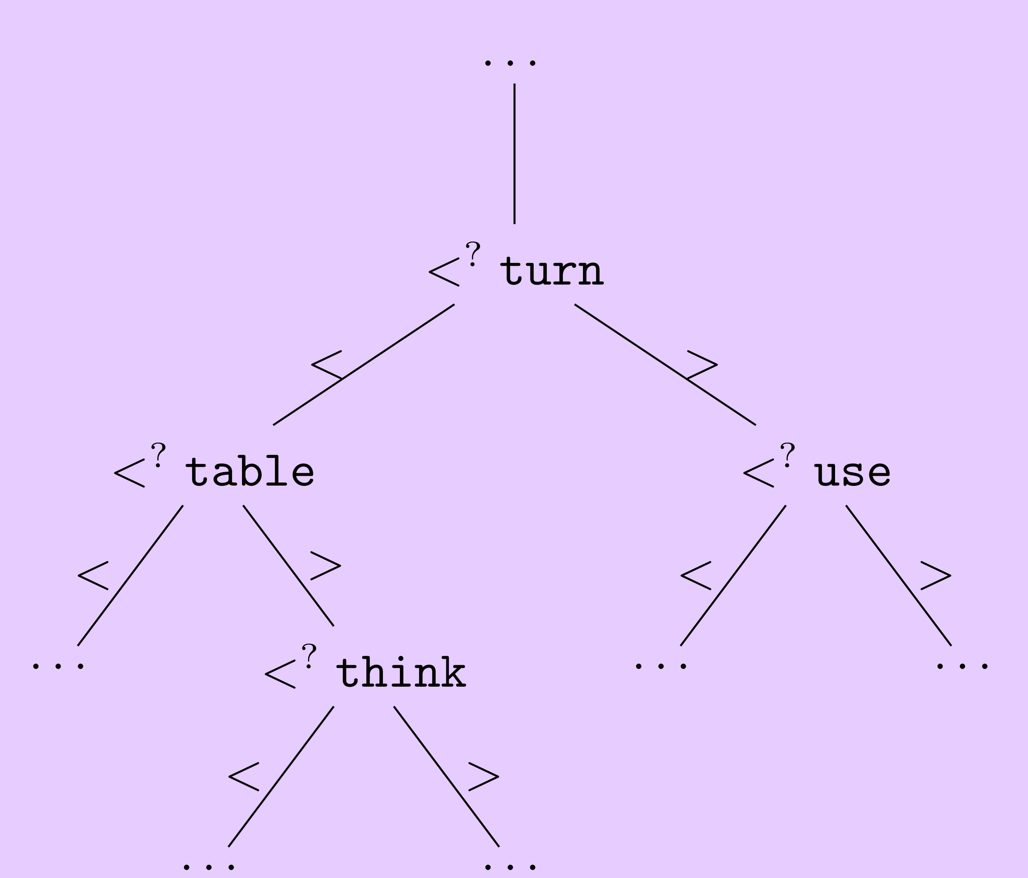
Analogy to a Word-Guessing Game

Inspiration for the coding technique:

> "*Is the secret word lexicographically smaller than w?*"

Translation into string-processing terms:

> "*Is the (infinite) input string lexicographically smaller than w?*"



Technical Tools

Input alphabet:

$$\Sigma \triangleq \{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$$

Output alphabet:

$$\mathbf{2} \triangleq \{\mathtt{0}, \mathtt{1}\}$$

Lexicographic bounds:

$$\mathcal{B} \triangleq \{\epsilon\} \cup \Sigma^* \cdot (\Sigma - \{\mathtt{a}\}) \cup \{\infty\}$$

Plain and split lexicographic intervals:

$$\mathcal{I} \triangleq \{[r,\, t] \mid r, t \in \mathcal{B} \text{ and } r < t\}$$
$$\widehat{\mathcal{I}} \triangleq \{[r \,\langle s\rangle\, t] \mid r, s, t \in \mathcal{B} \text{ and } r < s < t\}$$

Contents of intervals, by extension:

$$\mathrm{X}([r,\, t]) = \mathrm{X}([r\,\langle s\rangle\, t]) \triangleq \{\omega \in \Sigma^\infty \mid r < \omega < t\}$$

Conversion to plain intervals:

$$\mathrm{U}([r\,\langle s\rangle\, t]) \triangleq [r,\, t]$$

Conversion to split intervals:

$$\mathrm{S}_{\widehat{\mathcal{F}}}([r,\, t]) \triangleq \left\{[r\,\langle s\rangle\, t] \in \widehat{\mathcal{F}}\right\}$$

Trimming of lexicographic intervals:

$$\begin{array}{lcll}
\mathrm{T}([\epsilon, \mathtt{a}\cdot w]) & = & \mathrm{T}([\epsilon, w]) & \\
\mathrm{T}([\epsilon, \mathtt{b}]) & = & [\epsilon, \infty] & \\
\mathrm{T}([\epsilon, d\cdot w]) & = & [\epsilon, d\cdot w], & \text{if } (d = \mathtt{b} \text{ and } w \neq \epsilon) \text{ or } d > \mathtt{b} \\
\mathrm{T}([\epsilon, \infty]) & = & [\epsilon, \infty] & \\
\mathrm{T}([c\cdot v, c\cdot w]) & = & \mathrm{T}([v, w]) & \\
\mathrm{T}([c\cdot v, d]) & = & \mathrm{T}([v, \infty]), & \text{if } \mathrm{succ}(c) = d \\
\mathrm{T}([c\cdot v, d\cdot w]) & = & [c\cdot v, d\cdot w], & \text{if } (\mathrm{succ}(c) = d \text{ and } w \neq \epsilon) \text{ or } \mathrm{succ}(c) < d \\
\mathrm{T}([c\cdot v, \infty]) & = & [c\cdot v, \infty], & \text{if } c < \mathtt{z} \\
\mathrm{T}([\mathtt{z}\cdot v, \infty]) & = & \mathrm{T}([v, \infty]) &
\end{array}$$

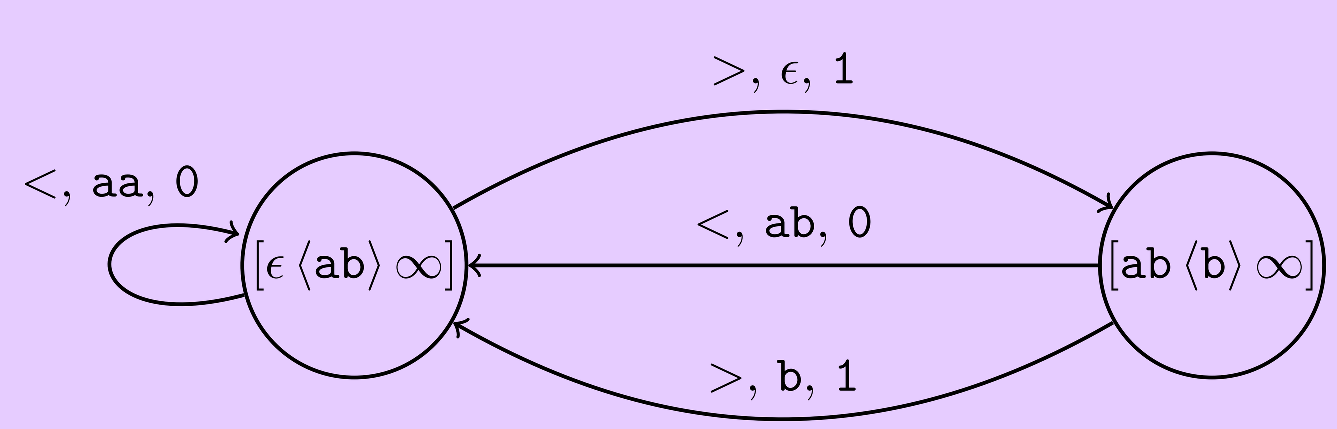
Effect of the trimming operation:

$$\mathrm{X}([r,\, t]) = \{w\} \cdot \mathrm{X}(\mathrm{T}([r,\, t])), \qquad \text{where } w \text{ is the LCP of } \mathrm{X}([r,\, t])$$

Coding rate of an automaton, in source symbols per output bit:

$$R(\widehat{\mathcal{Q}}) = \sum_{[r\,\langle s\rangle\, t] \in \widehat{\mathcal{Q}}} \begin{pmatrix} |\mathrm{LCP}(\mathrm{X}([r,\, s]))| \cdot \Pr(r < \Omega < s \mid r < \Omega < t) \\ +|\mathrm{LCP}(\mathrm{X}([s,\, t]))| \cdot \Pr(s < \Omega < t \mid r < \Omega < t) \end{pmatrix} \cdot \hat{p}([r\,\langle s\rangle\, t])$$

Example of an Automaton



Automaton: $\widehat{\mathcal{Q}} = \{[\epsilon\,\langle\mathtt{ab}\rangle\,\infty],\, [\mathtt{ab}\,\langle\mathtt{b}\rangle\,\infty]\}$
Probs.: $p(\mathtt{a}) = 0.7$ and $p(\mathtt{b}) = 0.3$
Coding rate: $R(\widehat{\mathcal{Q}}) = 1.126$ sym./bit

Definition of Automaton

An automaton is defined as a set of split intervals $\widehat{\mathcal{Q}} \subset \widehat{\mathcal{I}}$ with the following properties. Let $\mathcal{Q} = \{\mathrm{U}(I) \mid I \in \widehat{\mathcal{Q}}\}$.

Finiteness Set $\widehat{\mathcal{Q}}$ is finite.

Existence of a start state There exists a start state; i.e. $[\epsilon,\, \infty] \in \mathcal{Q}$.

Closure Transitions always lead to other states in $\widehat{\mathcal{Q}}$; i.e. for any $[r\,\langle s\rangle\, t] \in \widehat{\mathcal{Q}}$, we have both $\mathrm{T}([r,\, s]) \in \mathcal{Q}$ and $\mathrm{T}([s,\, t]) \in \mathcal{Q}$.

Determinism Given a specific knowledge about the input string, the automaton systematically decides to apply the same test on the input string; i.e. for any $[r,\, t] \in \mathcal{Q}$, there exists $s \in \mathcal{B}$ such that $\mathrm{S}_{\widehat{\mathcal{Q}}}([r,\, t]) = \{[r\,\langle s\rangle\, t]\}$.

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