Robust Transmission over Channels with Channel Uncertainty: An Algorithmic Perspective

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IEEE International Conference on Acoustics, Speech and Signal Processing

May 2020

Motivation

- Provision of accurate CSI is a major challenge in wireless systems due to
  - dynamic nature of the wireless channel
  - estimation inaccuracy
  - limited feedback
  - ...

➠ Imperfect CSI must be taken into account in the system design

- We consider the general uncertainty model of *compound channels*

➠ Capacity is known, but optimal signal processing and coding schemes remain unknown in general

- Such optimal schemes have been found only for very few specific cases and accordingly, common belief is that it is a hard problem to find them

In this work, we shed some new light upon this issue by adopting an *algorithmic perspective*

Overview Main Results

- We address this issue from a fundamental algorithmic point of view by using the concept of a *Turing machine* and the corresponding *computability framework*

➠ We study algorithmic computability of the capacity

Perfect CSI

Capacity of *discrete memoryless channels (DMCs)* is computable:

$$C(W) \in \mathbb{R}_c$$

for computable $W \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y})$.

Imperfect CSI

Capacity of *compound channels (CCs)* is in general non-computable:

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Birth of Information Age

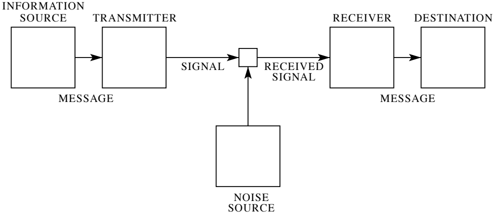
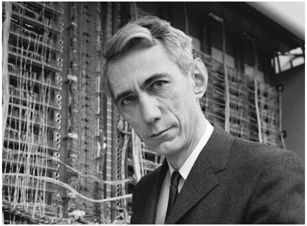


Fig. 1 — Schematic diagram of a general communication system.



- Claude Shannon laid the theoretical foundations for information theory, a mathematical communication model
  - ➠ **A mathematical theory of communication**

C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, no. 3, pp. 379–423, Jul. 1948

Perfect Channel State Information

- Discrete memoryless channels (DMCs)
- Let $\mathcal{X}$ and $\mathcal{Y}$ with $|\mathcal{X}| < \infty$ and $|\mathcal{Y}| < \infty$ be finite input and output alphabets
- Probability law for DMCs is specified by the channel

$$W^n(y^n|x^n) = \prod_{i=1}^{n} W(y_i|x_i)$$

➠ Belong to the class of independent and identically distributed (i.i.d.) channels which represent the most tractable class of channel laws

The *capacity* $C(W)$ of a discrete memoryless channel (DMC) W is

$$\boldsymbol{C(W) = \max_{X} I(X;Y) = \max_{p \in \mathcal{P}(\mathcal{X})} I(p,W)}$$

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- *Entropic quantities*
- *Single-letter*
- *Convex optimization problem*

- Of particular relevance as it allows to compute the capacity $C(W)$ as a function of the channel W given by a convex optimization problem

Can we compute the capacity algorithmically?

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1936: Birth of Computer Science

- Alan M. Turing was the first to study this kind of problems systematically
- He developed a computing model
  - ➠ **Turing machine**
- Object of interest: real numbers

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

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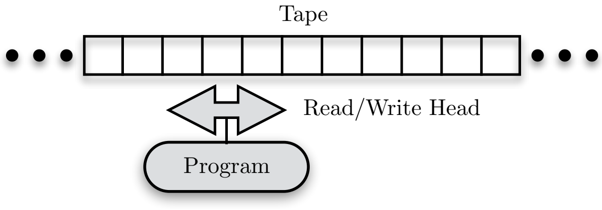
Turing Machine: The Most Powerful Computation Model



Mathematical model of an abstract machine that manipulates symbols on a strip of tape according to certain given rules

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

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Turing Machine (2)

Turing machines can simulate any given algorithm and therewith provide a simple but very powerful model of computation

- **No** limitation on computational complexity
- Unlimited computing capacity and storage
- Completely error-free execution of programs
- Most powerful programming languages are Turing-complete (such as C, C++, Java, etc.)
- All discrete computing models are equivalent
  (von Neumann, Gödel, Minsky, . . .)

Any arbitrarily large finite-dimensional problem can be exactly solved without errors by a Turing machine

Turing Machine (3)

Turing machines are suited to study the limitations in performance of a digital computer:

Anything that is not Turing computable cannot be computed on a real digital computer, regardless of how powerful it may be

- Alan Turing introduced the concept of a computable real number in 1936, and demonstrated some principal limitations of computability
- In 1949 a computable monotonically increasing sequence which converges to a real non-computable number was constructed

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Computability of Numbers

Computable numbers are real numbers that are computable by Turing machines

Exact definition:

- A sequence $\{r_n\}_{n\in\mathbb{N}}$ is called a computable sequence if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}$$

- A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ such that

$$|x - r_n| < 2^{-n}$$

Key idea: effective approximation

- $\mathbb{R}_c$ computable real numbers
- Commonly used constants like e and π are computable

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Computability of Distributions and Channels

- Based on this, we can define *computable probability distributions* and *computable channels*

- We define the set of computable probability distributions $\mathcal{P}_c(\mathcal{X})$ as the set of all probability distributions

$$p \in \mathcal{P}(\mathcal{X}) \text{ such that } p(x) \in \mathbb{R}_c,\ x \in \mathcal{X}$$

- Let $\mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ be the set of all computable channels, i.e., for a channel

$$W : \mathcal{X} \to \mathcal{P}(\mathcal{Y}) \text{ we have } W(\cdot|x) \in \mathcal{P}_c(\mathcal{Y}) \text{ for every } x \in \mathcal{X}$$

Computability of $C(W)$

- *Warm-up:* Let's see if for a computable channel $W \in \mathcal{CH}_c$ the capacity $C(W)$ is computable...

Theorem:

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then for all computable channels $W \in \mathcal{CH}_c$ we have

$$C(W) = \max_{p \in \mathcal{P}(\mathcal{X})} I(p, W) \in \mathbb{R}_c.$$

➠ The **capacity** $C(W)$ for a computable channel $W \in \mathcal{CH}_c$ is computable and **can be algorithmically computed by a Turing machine**!

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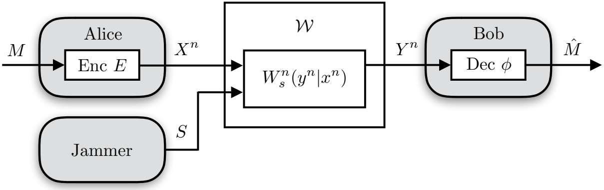
Channel Uncertainty



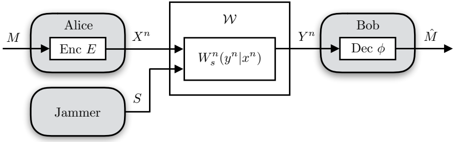
- Due to the nature of the wireless channel, but also due to implementation issues such as estimation and feedback inaccuracy, practical systems suffer from channel uncertainty
- Let $\mathcal{S}$ be an arbitrary state (uncertainty) set
- State $s \in \mathcal{S}$ is unknown, but remains *constant* throughout the transmission

The *compound channel*

$$\mathcal{W} := \{W_s \in \mathcal{CH}(\mathcal{X};\mathcal{Y}) : s \in \mathcal{S}\}$$

is given by the collection of all channels $W_s \in \mathcal{CH}(\mathcal{X};\mathcal{Y})$ for all states $s \in \mathcal{S}$.

Compound Channel



The *capacity* $C(\mathcal{W})$ of a compound channel (CC) $\mathcal{W}$ is

$$C(\mathcal{W}) = \sup_{p \in \mathcal{P}(\mathcal{X})} \inf_{s \in \mathcal{S}} I(p, W_s)$$

- *Analytically* well understood (closed-form single letter entropic expression)
- Surprisingly, not much known about its algorithmic computability and the optimal signal processing

➠ Study its structure and algorithmic computability of optimal strategies

D. Blackwell, L. Breiman, and A. J. Thomasian, "The capacity of a class of channels," *Ann. Math. Stat.*, vol. 30, no. 4, pp. 1229–1241, Dec. 1959

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Computability of $C(\mathcal{W})$

A compound channel $\mathcal{W} = \{W_s \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y}) : s \in \mathcal{S}\}$ is said to be *computable* if there is a recursive function $\varphi : \mathcal{S} \to \mathcal{CH}_c$ with $\varphi(s) = W_s$ for all $s \in \mathcal{S}$. The set of all computable compound channels is denoted by $\mathcal{CC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$.

- Compound set $\mathcal{W}$ is algorithmically constructible, i.e., for every state $s \in \mathcal{S}$ the channel W_s can be constructed by an algorithm with input s

Theorem:

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then there is a computable compound channel $\mathcal{W} \in \mathcal{CC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$ such that

$$C(\mathcal{W}) = \sup_{p\in\mathcal{P}(\mathcal{X})} \inf_{s\in\mathcal{S}} I(p, W_s) \notin \mathbb{R}_c.$$

- Although the channel itself is computable, i.e., $\mathcal{W} \in \mathcal{CC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$, it is not possible to algorithmically compute $C(\mathcal{W})$!

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Conclusions

- Computability framework based on **Turing machines**
- *Computability of capacities*
  - Capacity value of DMCs is computable: $C(W) \in \mathbb{R}_c$
  - Capacity value of CCs is in general **not** computable: $C(\mathcal{W}) \notin \mathbb{R}_c$
- *Further extensions and consequences*
  - Optimal signal processing and coding schemes are not effectively constructible
  - CSIT or CSIR do not change this behavior and capacity value remains **non**-computable in general: $C_{\mathsf{CSIT}}(\mathcal{W}) \notin \mathbb{R}_c$ & $C_{\mathsf{CSIR}}(\mathcal{W}) \notin \mathbb{R}_c$
  - Finite block length performance cannot be characterized by computable functions

Thank you for your attention!

Supported in part by

Federal Ministry of Education and Research

Molecular Communication (MAMOKO) – 16KIS0914
Post Shannon Communication (NewCom) – 16KIS1004

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Gottfried Wilhelm Leibniz Programme – BO 1734/20-1
Germany's Excellence Strategy – EXC-2111 – 390814868

NSF

CCF-0939370, CCF-1513915, and CCF-1908308

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