Node-Asynchronous Spectral Clustering on Directed Graphs

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California Institute of Technology

45th International Conference on
Acoustics, Speech and Signal Processing
(ICASSP 2020)

Caltech

Outline

1. Graph Signal Processing and Spectral Clustering
2. Asynchronous Updates on Graphs
3. Autonomous Spectral Clustering
4. Conclusion

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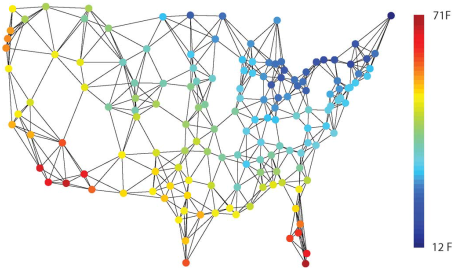
Preliminaries



$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{C}^N$$

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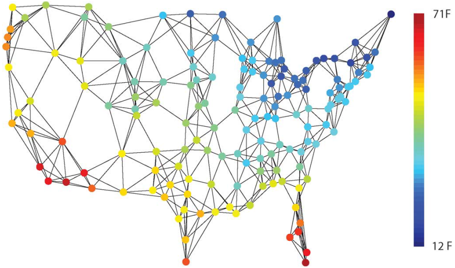

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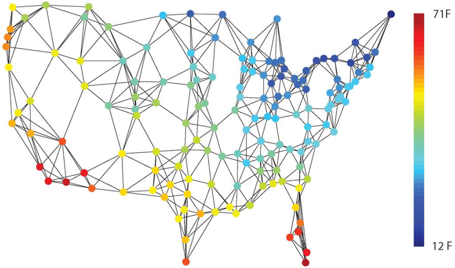

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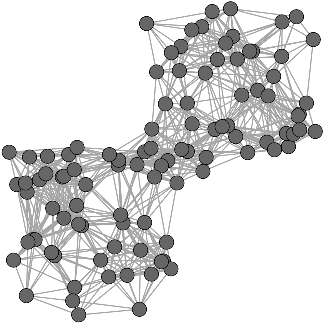
Adjacency matrix[1] : $\mathbf{A}$
Graph Laplacians[2] : $\mathbf{L}$, or $\mathcal{L}$
Other selections[3]

[1] Sandryhaila & Moura, "Discrete Signal Processing on Graphs," *IEEE Trans. S. P. vol. 61, no. 7, 2013*

[2] Shuman et al, "The emerging field of signal processing on graphs: ...," *IEEE S. P. Magazine, vol. 30, no. 3 2013*

Spectral Clustering



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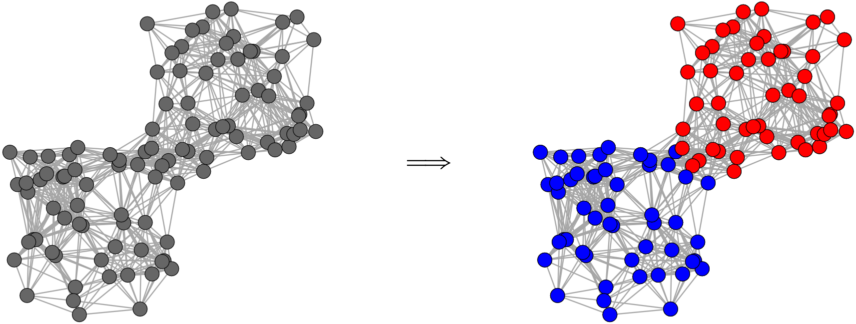

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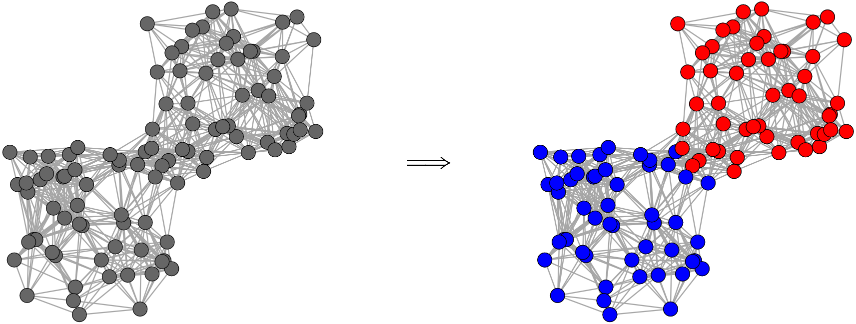

Graph Laplacian: $\mathbf{L} = \mathbf{D} - \mathbf{A}$

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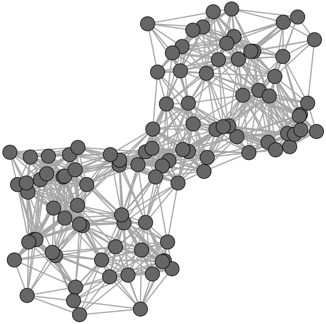
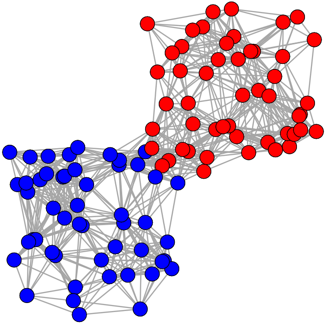


$\Longrightarrow$



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$0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$
(Fiedler Value)

$\mathbf{L}\,\mathbf{v}_2 = \lambda_2\,\mathbf{v}_2$
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[1] Fiedler, "Algebraic connectivity of graphs," *Czechoslovak mathematical journal, 1973*

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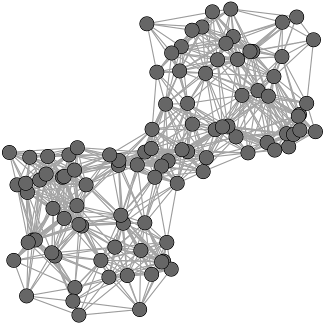

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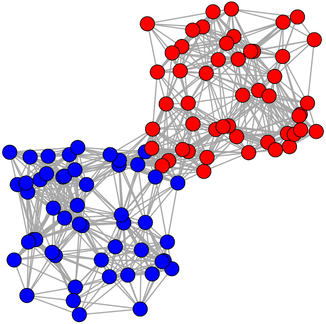

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Compute $\mathbf{v}_2$ with random asynchronous computations?

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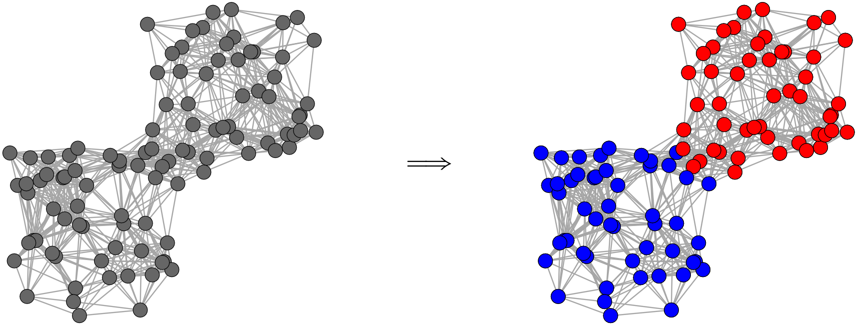
Spectral Clustering *Here: Directed Case*



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$\mathbf{x}_k$ = Signal on the Graph

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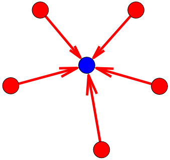

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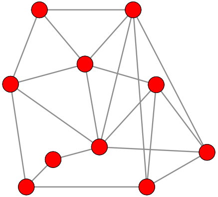

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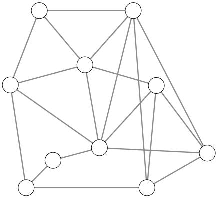

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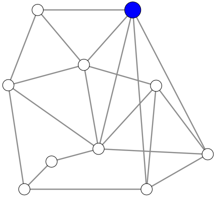

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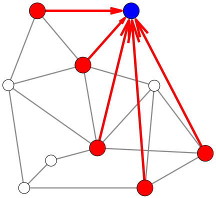

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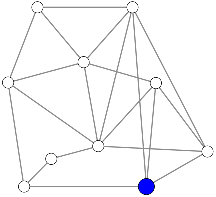

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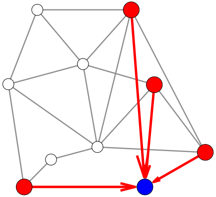

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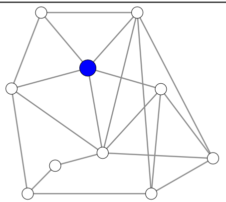

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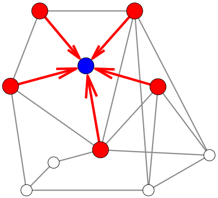

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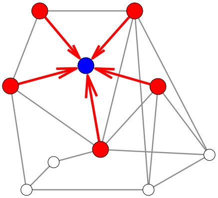

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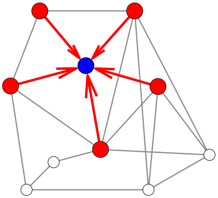

Recurrent NN
(Hopfield Model)[1]

$$x_k[i] = \theta\left(\mathbf{a}_i\ \mathbf{x}_{k\text{-}1} + u_i\right)$$

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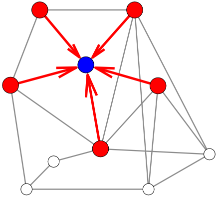

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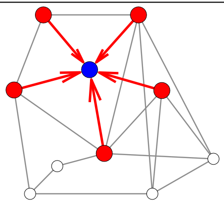

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Synchronous case: $\rho(\mathbf{A}) < 1 \implies \lim_{k\to\infty} \mathbf{x}_k = (\mathbf{I} - \mathbf{A})^{\text{-}1}\,\mathbf{u}$

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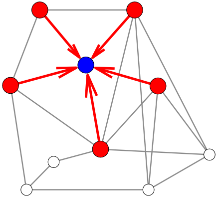

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Random Asynchronous case: To be discussed next ...

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$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{u} \quad \Rightarrow \quad \boxed{\mathbf{u} \in \text{range}(\mathbf{I} - \mathbf{A})}$$

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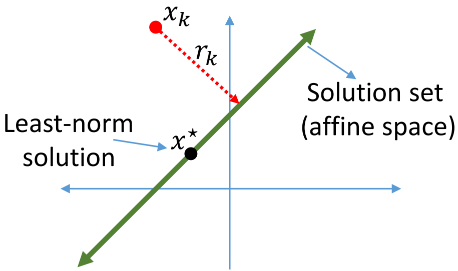

Mean-Squared Convergence of the Updates

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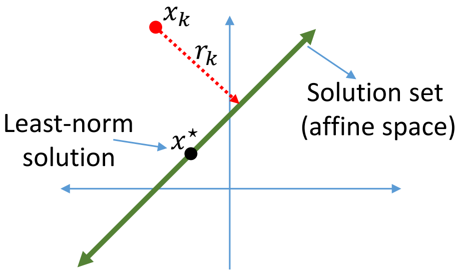

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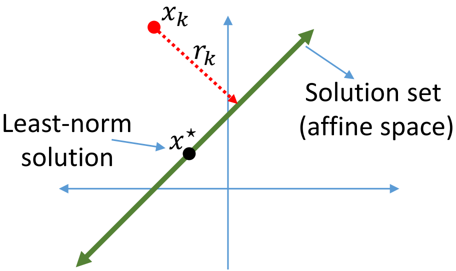

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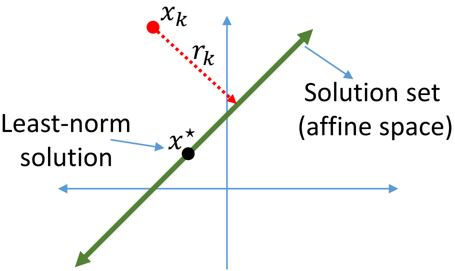

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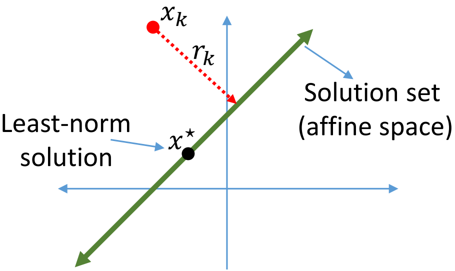

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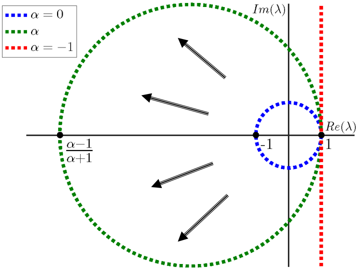

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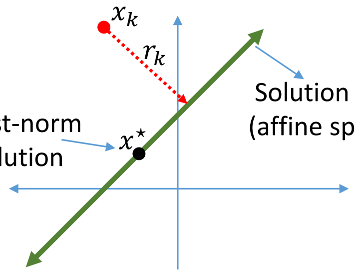

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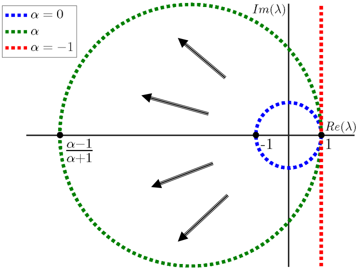

$$\Downarrow$$

$$\lim_{k\to\infty} \mathbb{E}\big[\|\mathbf{r}_k\|_2^2\big] = 0$$

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Mean-Squared Convergence of the Updates - Cont.

Theorem (The necessary and sufficient condition)

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Computing Eigenvectors

$$\mathbf{u} = \mathbf{0} \qquad \Longrightarrow \qquad x_k[i] = \begin{cases} \mathbf{a}_i\ \mathbf{x}_{k\text{-}1}, & \text{w.p.} \quad p_i, \\ x_{k\text{-}1}[i], & \text{w.p.} \quad 1\text{-}p_i. \end{cases}$$

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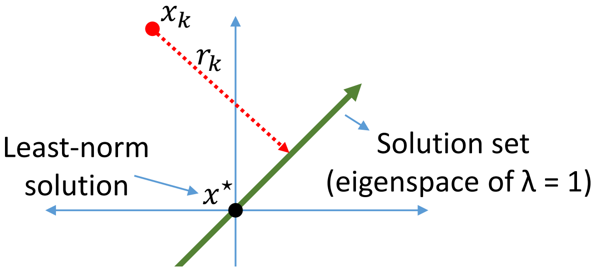

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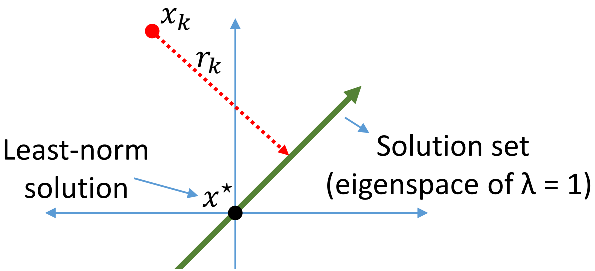

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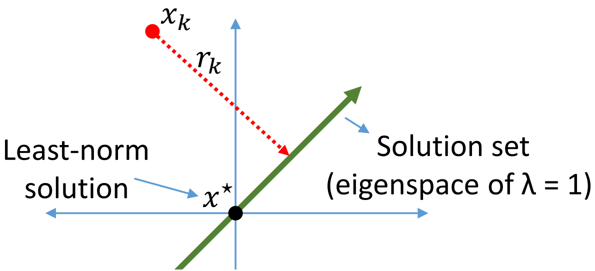

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$$\lim_{k\to\infty} \mathbb{E}\left[\|\mathbf{r}_k\|_2^2\right] = \mathbf{0}$$

$$\Updownarrow$$

$\mathbf{x}_k$ converges to
an eigenvector of $\lambda = 1$

Outline

1 Graph Signal Processing and Spectral Clustering

2 Asynchronous Updates on Graphs

3 Autonomous Spectral Clustering

4 Conclusion

Use of Graph Polynomials

$$\mathbf{A}$$

Asynchronous

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$\lambda = 1$

$\mathbf{A}$

Asynchronous

Use of Graph Polynomials

$$\lambda = 1 \quad \xrightarrow[\text{Asynchronous}]{\mathbf{A}} \quad \lim_{k\to\infty} \mathbf{x}_k \in \text{null}(\mathbf{A} - \mathbf{I})$$

Use of Graph Polynomials

$\lambda = 1$ $\xrightarrow[\text{Asynchronous}]{\mathbf{A}}$ $\lim_{k \to \infty} \mathbf{x}_k \in \text{null}(\mathbf{A} - \mathbf{I})$

$$H(\mathbf{A}) = \sum_{k=0}^{L} h_k \, \mathbf{A}^k$$

Asynchronous

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(Fiedler Vector)

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L^{th} order $\Rightarrow$ L-hop neighborhood

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$$\mathbf{\Phi} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^L \\ 1 & \lambda_2 & \cdots & \lambda_2^L \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^L \end{bmatrix}$$

$$\mathbf{h} = [h_0 \;\; h_1 \;\; \cdots \;\; h_L]^T$$

Use of Graph Polynomials

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$$\max_{\mathbf{h}} \quad c \quad \text{s.t.} \quad \begin{array}{c} \phi_2 \, \mathbf{h} = 1 \\ |\bar{\mathbf{\Phi}} \, \mathbf{h}| \leqslant (1 - c) \, \mathbb{1}_{N\text{-}1} \end{array}$$

$$c \geqslant 0$$

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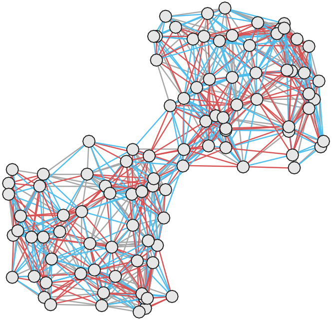
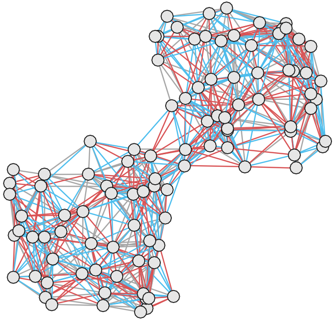
$L = 2$ *works in practice*

A Numerical Application



A Numerical Application $(\mathbf{L})$



$$0 = \lambda_1 < |\lambda_2| \leqslant \cdots \leqslant |\lambda_N|$$

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[1] Fiedler, "Algebraic connectivity of graphs," *Czechoslovak mathematical journal, 1973*

[2] Zhou, Huang, and Scholkopf, "Learning from labeled and unlabeled data on a directed graph," *NIPS, 2005*

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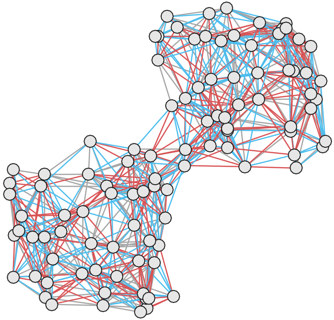

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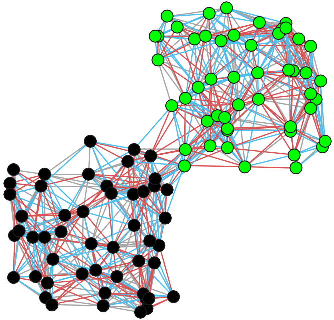

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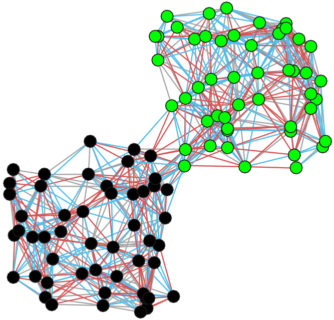
(L)



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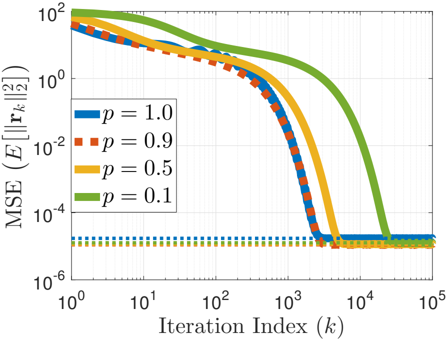

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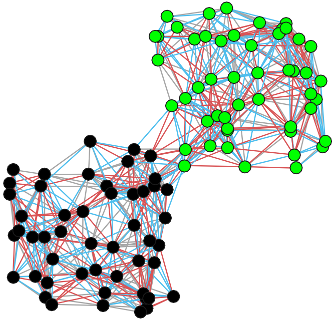
A Numerical Application (L)



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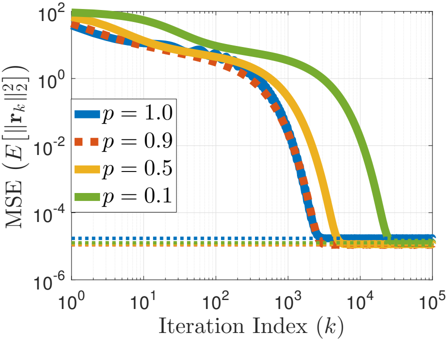

p : Update probabilities

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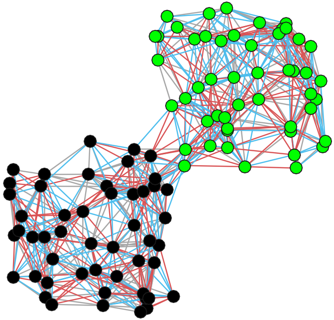

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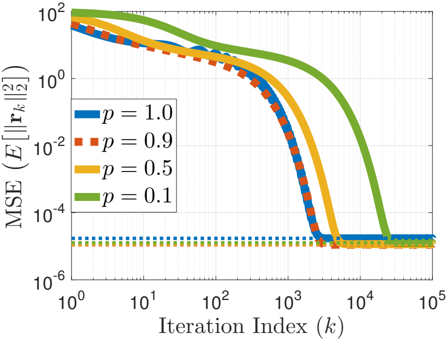

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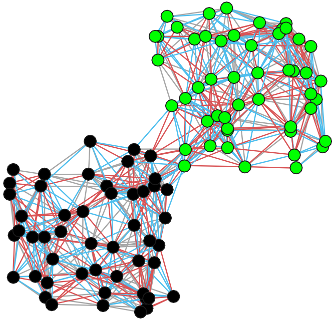

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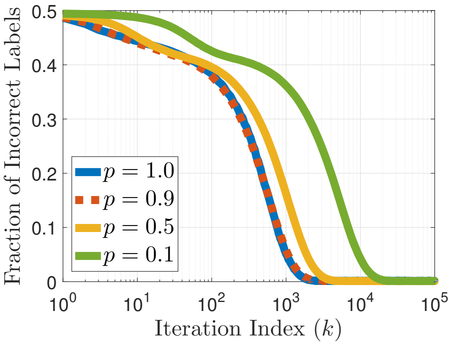

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Outline

1 Graph Signal Processing and Spectral Clustering

2 Asynchronous Updates on Graphs

3 Autonomous Spectral Clustering

4 Conclusion

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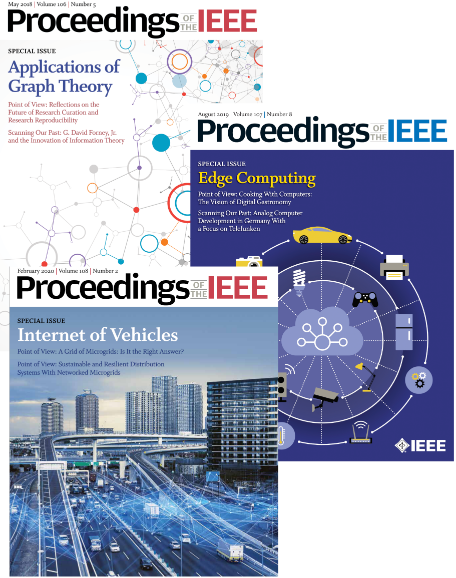
Thank you!

New Horizons ...



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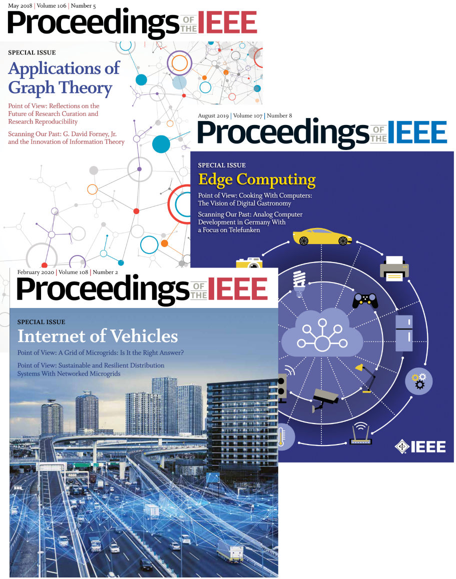

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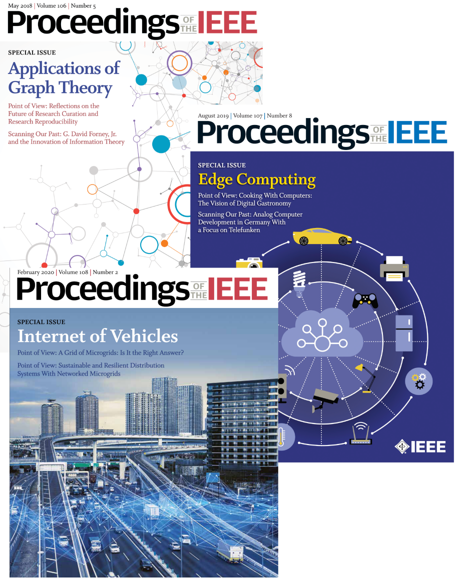

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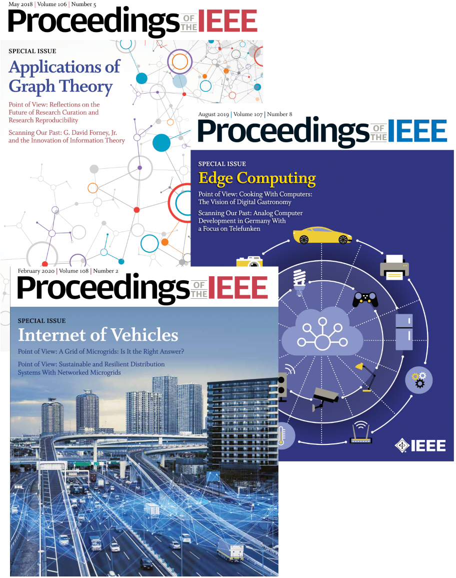

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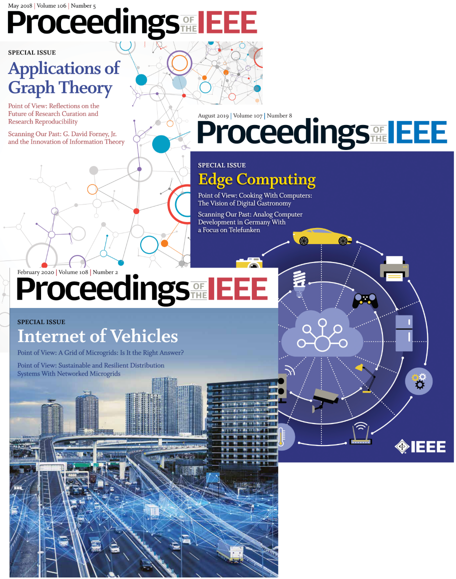

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