Diagonalizable Shift and Filters for Directed Graphs

based on the Jordan-Chevalley Decomposition

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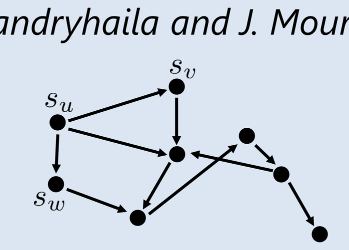
Goal

Digraph Signal Processing (DGSP):
Signals indexed by nodes of directed graphs
[A. Sandryhaila and J. Moura, 2013]



$$A = \begin{bmatrix} 0&1&1&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

Shift = adjacency matrix A

Filters = polynomials in A

Filter Algebra $\langle A\rangle \cong \mathbb{C}[x]/m_A(x)$

Problem: In general, A may not be diagonalizable

Our work: Diagonalizable Graph Shift:

- Identify sub-algebra of diagonalizable filters
- Compute associated diagonalizable shift A_D

$$\langle A_D\rangle \le \langle A\rangle$$

Digraph Signal Processing

Directed Graph

$$G(V,E,A)$$

adjacency matrix with weights

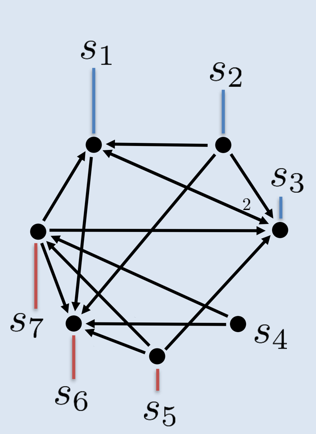
Graph Signal

$$s : V \to \mathbb{C};\, v \mapsto s_v$$

Graph Shift

$$A : \mathbb{C}^n \to \mathbb{C}^n;\, \mathbf{s} \mapsto A\mathbf{s}$$

Example: Signal on Digraph



$$A = \begin{bmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&1\\ 0&0&1&0&0&1&1\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&1&0 \end{bmatrix}$$

$\mathbf{s} = \begin{pmatrix} 3 & 2 & 1 & 0 & -1 & -2 & -3 \end{pmatrix}^T$ Signal

$A\mathbf{s} = \begin{pmatrix} 0 & 2 & 3 & -5 & -4 & 0 & 2 \end{pmatrix}^T$ *Shifted* signal

Graph Fourier Transform

$$F : \mathbb{C}^n \to \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{g_i} S_{ij}$$

pure frequencies (Jordan subspaces)

Matrix Representation $F = V^{-1}$ where $A = VJV^{-1}$

Digraph Signal Processing

Graph Filters

$$H : \mathbb{C}^n \to \mathbb{C}^n;\, H = p(A) = \sum_{i=0}^{d-1} h_i A^i \qquad \text{(Polynomial in } A\text{)}$$

$$H(As) = AH(s) \qquad \text{(Shift-Invariance)}$$

Filter Algebra $\langle A\rangle \cong \mathbb{C}[x]/m_A(x)$

Frequency Response

Shift's response $FAF^{-1} = J = \bigoplus_i J_{d_i}(\lambda_i)$

Jordan Block

Size of block

$$FHF^{-1} = Fh(A)F^{-1} = h(J) = \bigoplus_i h(J_{d_i}(\lambda_i))$$

$$h(J_d(\lambda)) = \begin{pmatrix} h(\lambda) & \frac{h'(\lambda)}{1!} & \dots & \frac{h^{(d-1)}(\lambda)}{(d-1)!} \\ 0 & h(\lambda) & \dots & \frac{h^{(d-1)}(\lambda)}{(d-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h(\lambda) \end{pmatrix}$$

Computation of polynomial evaluated on Jordan Block

Diagonalizable Shift and Filters

Theorem *(Jordan–Chevalley)*

Any matrix A can be decomposed uniquely as
$$A = A_D + A_N$$

- A_D is diagonalizable and A_N is nilpotent
- A_D and A_N are polynomials in A
- $A_D A_N = A_N A_D$

Consequences for DGSP

Diagonalizable shift

$$\langle A_D\rangle \le \langle A\rangle$$

is the sub-algebra of all diagonalizable filters

How do we compute A_D *?*

Solution: Hermite Interpolation

For all eigenvalues λ_i

$$p(\lambda_i) = \lambda_i, p'(\lambda_i) = 0, \dots, p^{(d_i-1)}(\lambda_i) = 0 \qquad (1)$$

$$\tilde{p}(\lambda_i) = \lambda_i, \tilde{p}'(\lambda_i) = 0, \dots, \tilde{p}^{(a_i-1)}(\lambda_i) = 0 \qquad (2)$$

d_i is the dimension of the largest Jordan block of λ_i
a_i is algebraic multiplicity of λ_i

$$\tilde{p}(x) \equiv p(x) \mod m_A(x)$$

$J_D = \tilde{p}(J) = p(J)$ is the diagonal part of J

Diagonalizable Shift

$$A_D = \tilde{p}(A) = p(A) = p(VJV^{-1}) = VJ_DV^{-1}$$

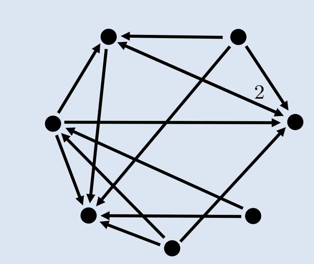
Properties of Diagonalizable Shift

- FA_DF^{-1} is the diagonal part J_D of J
- $m_{A_D}(x) = (x-\lambda_1)\dots(x-\lambda_k)$
- Sub-algebra of diagonalizable filters is

$$\langle A_D\rangle \cong \mathbb{C}[x]/m_{A_D}(x)$$

Example

Directed Graph



$$A = \begin{bmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&1\\ 0&0&1&0&0&1&1\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&1&0 \end{bmatrix}$$

Frequency Response of Shift

$$FAF^{-1} = J = \begin{array}{c} \overbrace{\qquad\qquad}^{d_1} \qquad\qquad \overbrace{\quad}^{d_2}\ \overbrace{\quad}^{d_3} \\ \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-\sqrt{2}&0\\ 0&0&0&0&0&0&\sqrt{2} \end{bmatrix} \\ \underbrace{\qquad\qquad\qquad\qquad}_{a_1}\ \underbrace{\quad}_{a_2}\ \underbrace{\quad}_{a_3} \end{array}$$

$m_A(x) = x^3(x+\sqrt{2})(x-\sqrt{2})$ *minimal polynomial*

$\chi_A(x) = x^5(x+\sqrt{2})(x-\sqrt{2})$ *characteristic polynomial*

Filter Algebra $\langle A\rangle \cong \mathbb{C}[x]/x^3(x^2-2)$

Diagonalizable Shift Computation

Hermite Interpolation yields

$$p(x) = \frac{1}{2}x^3$$

$$\tilde{p}(x) = \frac{1}{4}x^5 \equiv \frac{1}{2}x^3 = p(x) \mod x^3(x^2-2)$$

Diagonalizable Shift

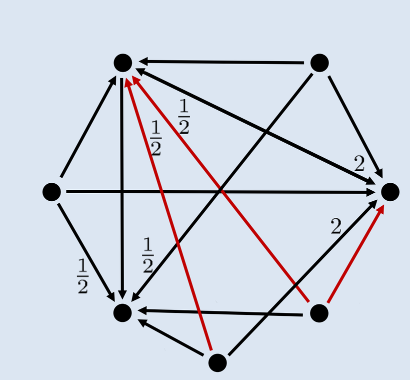
$$J_D = \tfrac{1}{4}J^5 = \tfrac{1}{2}J^3 = \mathrm{diag}(0,0,0,0,0,-\sqrt{2},\sqrt{2})$$

$$A_D = \tfrac{1}{4}A^5 = \tfrac{1}{2}A^3 = \begin{bmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&\frac{1}{2}&0\\ 1&0&0&0&0&0&0\\ \frac{1}{2}&0&1&0&0&1&0\\ \frac{1}{2}&0&2&0&0&1&0\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&\frac{1}{2}&0 \end{bmatrix}$$

Jordan – Chevalley Decomposition

$$A = A_D + A_N = \begin{bmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&\frac{1}{2}&0\\ 1&0&0&0&0&0&0\\ \frac{1}{2}&0&1&0&0&1&0\\ \frac{1}{2}&0&2&0&0&1&0\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&\frac{1}{2}&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2}&0\\ 0&0&0&0&0&0&0\\ -\frac{1}{2}&0&-1&0&0&0&1\\ -\frac{1}{2}&0&-1&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2}&0 \end{bmatrix}$$

Graph associated with Shift

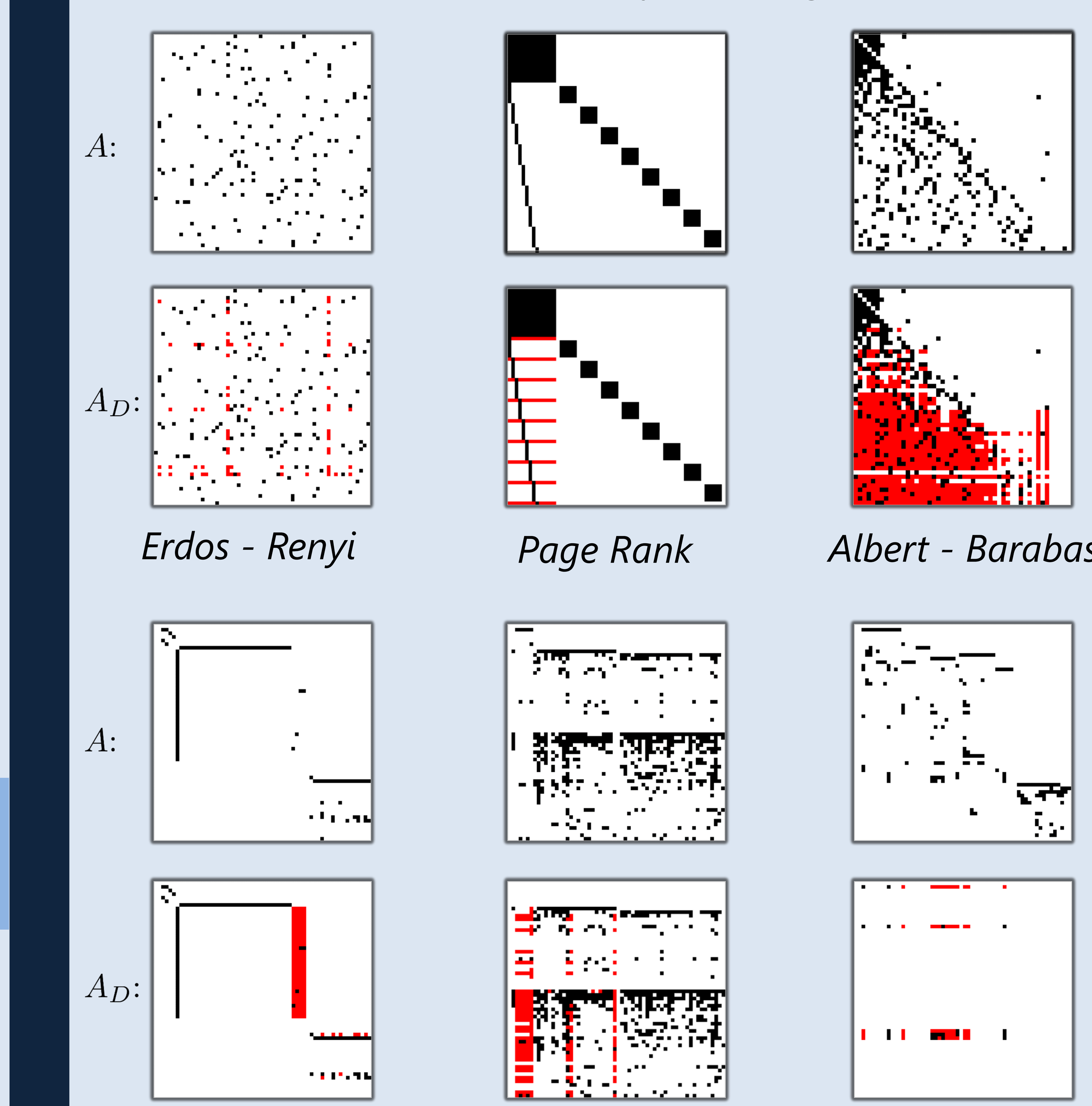


red : added edges compared to A

Experiments

Artificial and Real Graphs

Black Entries: Original edges
Red Entries: Newly added edges



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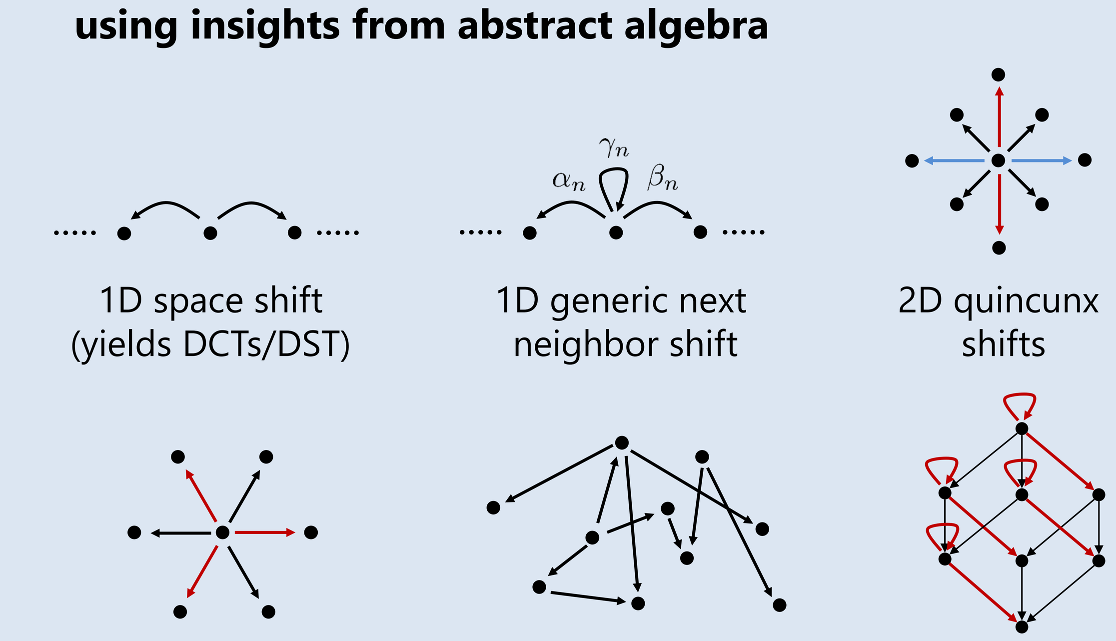
Remarks

- A_D provides DGSP using a sub-algebra of diagonalizable filters
- The amount of edges added by A_D depends on the complexity of the Jordan Normal Form.

Algebraic Signal Processing (ASP)

Framework to generalize standard, linear, time DSP using insights from abstract algebra



1D space shift (yields DCTs/DST)
1D generic next neighbor shift
2D quincunx shifts
2D hexagonal shifts
graph shift (yields graph DSP)
set shifts

http://www.ece.cmu.edu/~smart/
https://acl.inf.ethz.ch/research/ASP/