Computability of the Peak Value of Bandlimited Signals

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Motivation

Peak value problem: The task of computing the peak value of a bandlimited signal from its samples.

- Important for applications, e.g., in communications (OFDM), where the peak value of the transmit signal has to be controlled.
- Often, the signals are created in the digital domain (digital baseband signal) and then converted into the analog domain (transmit signal).
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Can we always compute the peak value of the analog signal algorithmically on digital hardware?

Overview of the Results

Computability of the peak value?	
Without oversampling	With oversampling
x No	✓ Yes
No algorithm exists	We give an algorithm
No control of the approximation error	Control of the approximation error

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- Consequences for the theory of communication systems.
- Textbook assumption: communication systems are sampled at Nyquist rate.

T. M. Cover and J. A. Thomas, *Elements of Information Theory*. John Wiley & Sons, 1991

R. G. Gallager, *Information Theory and Reliable Communication*. John Wiley & Sons, 1968

Turing Machine

Turing Machine:
Abstract device that manipulates symbols on a strip of tape according to certain rules.

- Turing machines are an idealized computing model.
- No limitations on computing time or memory, no computation errors.
- Although the concept is very simple, Turing machines are capable of simulating any given algorithm.

Turing machines are suited to study the limitations in performance of a digital computer:

> Anything that is not Turing computable cannot be computed on a real digital computer, regardless how powerful it may be.

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proceedings of the London Mathematical Society*, vol. s2-42, no. 1, pp. 230–265, Nov. 1936

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem. A correction," *Proceedings of the London Mathematical Society*, vol. s2-43, no. 1, pp. 544–546, Jan. 1937

Notation

- c_0: space of all sequences that vanish at infinity
  Norm: $\|x\|_{\ell^\infty} = \max_{k\in\mathbb{Z}}|x(k)|$
- $L^p(\Omega)$, $1 \leqslant p < \infty$: space of all measurable, pth-power Lebesgue integrable functions on Ω
  Norm: $\|f\|_p = \left(\int_\Omega |f(t)|^p \, dt\right)^{1/p}$
- $L^\infty(\Omega)$: space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite
- $f|_{\mathbb{Z}/a}$: sequence $\{f(k/a)\}_{k\in\mathbb{Z}}$ (restriction of f to the set $\mathbb{Z}/a = \{k/a\}_{k\in\mathbb{Z}}$)

Bandlimited Functions

Definition (Bernstein Space)

Let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leqslant C(\epsilon) \exp\big((\sigma + \epsilon)|z|\big)$ for all $z \in \mathbb{C}$.

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line.

- A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ.
- We have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r$ for all $1 \leqslant p \leqslant r \leqslant \infty$.
- $\mathcal{B}_{\sigma,0}^\infty$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.
- $\mathcal{B}_\sigma^2$: space of bandlimited functions with finite energy.

Computable Sequences of Rationals

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

- A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.

Computable Real Numbers

First example of an effective approximation

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|x - r_n| \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathbb{R}_c$: set of computable real numbers
- $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable.
- Commonly used constants like e and π are computable.

Computability in c_0

Computability in Banach spaces: Effective approximation by "simple" elements

A sequence $x = \{x(k)\}_{k \in \mathbb{Z}}$ in c_0 is called computable in c_0 if every number $x(k)$, $k \in \mathbb{Z}$, is computable and there exist a computable sequence $\{y_n\}_{n \in \mathbb{N}} \subset c_0$, where each y_n has only finitely many non-zero elements, all of which are computable as real numbers, and a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have

$$\|x - y_n\|_{\ell^\infty} \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- Effective approximation by simple / finite-length sequences
- $\mathcal{C}c_0$: set of all sequences that are computable in c_0

The Peak Value Problem I

Peak value problem:
If we know only the samples $f|_{\mathbb{Z}/a}$ of a continuous-time bandlimited signal $f \in \mathcal{B}^{\infty}_{\pi,0}$, can we determine $\|f\|_{\infty}$ (the peak value of f)?

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3. Which role plays the oversampling factor a?

The Peak Value Problem II

For all $f \in \mathcal{B}_{\pi,0}^{\infty}$ and $a > 1$, we have [BW02, WB03]

$$\|f\|_{\infty} \leqslant \frac{1}{\cos\left(\frac{\pi}{2a}\right)} \|f|_{\mathbb{Z}/a}\|_{\ell^{\infty}}.$$

- With oversampling, we can bound the peak value of f from above by an expression that uses only the peak value of the samples $f|_{\mathbb{Z}/a}$ and the oversampling factor a.

H. Boche and G. Wunder, “Über eine Verallgemeinerung eines Resultats von Riesz über trigonometrische Polynome auf allgemeine bandbegrenzte Funktionen,” *Zeitschrift für angewandte Mathematik und Mechanik (ZAMM)*, vol. 82, no. 5, pp. 347–351, May 2002

G. Wunder and H. Boche, “Peak value estimation of bandlimited signals from their samples, noise enhancement, and a local characterization in the neighborhood of an extremum,” *IEEE Trans. Signal Process.*, vol. 51, no. 3, pp. 771–780, Mar. 2003

Computation of the Peak Value With Oversampling

Theorem

Let $f \in \mathcal{B}_{\pi,0}^{\infty}$ *and* $a > 1$, $a \in \mathbb{R}_c$. *If* $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$ *then we have* $f(t) \in \mathbb{C}_c$ *for all* $t \in \mathbb{R}_c$*, and we can compute* $\|f\|_\infty \in \mathbb{R}_c$ *algorithmically.*

- If we know f on an oversampling set $\mathbb{Z}/a$, $a > 1$, then we can compute $\|f\|_\infty$ from the samples $f|_{\mathbb{Z}/a} = \{f(k/a)\}_{k \in \mathbb{Z}}$.
- In proof of the theorem an explicit algorithm is outlined.

Peak Value Without Oversampling

Theorem

For all $M \in \mathbb{N}$, there exists a signal $f_M \in \mathcal{B}^{\infty}_{\pi,0}$ such that $\|f_M|_{\mathbb{Z}}\|_{\ell^\infty} \leqslant 1$ and $\|f_M\|_\infty > M$.

- The peak value $\|f\|_\infty$ cannot be inferred from the norm of the samples $\|f|_{\mathbb{Z}}\|_{\ell^\infty}$.
- Thus, a result such as the upper bound $\frac{1}{\cos(\frac{\pi}{2a})}\|f|_{\mathbb{Z}/a}\|_{\ell^\infty}$ cannot hold when no oversampling is used.
- This result also implies that we cannot compute an upper bound of the peak value $\|f\|_\infty$ from the samples $f|_{\mathbb{Z}}$.

Computation of the Peak Value Without Oversampling

Without oversampling there exist signals for which it is not possible to compute the peak value.

Theorem

There exists a real-valued signal $f_1 \in \mathcal{B}^{\infty}_{\pi,0}$ *with* $f_1|_{\mathbb{Z}} \in \mathcal{C}c_0$ *such that* $\max_{t\in\mathbb{R}} f_1(t) \notin \mathbb{R}_c$.

Without oversampling the peak value of f_1 cannot be computed on any digital hardware, including DSPs, FPGAs, and CPUs.

Conclusions

- We analyzed the peak value problem for bandlimited signals with respect to computability.
- The peak value is computable on digital hardware if oversampling is used.
- We provided an algorithm that can be used to perform this computation.
- There exist signals for which the peak value problem cannot be algorithmically solved without oversampling.

Thank you!