Optimal Sampling Rate and Bandwidth of Bandlimited Signals — An Algorithmic Perspective

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Motivation

- The actual bandwidth $B(f)$ of a bandlimited signal is a key quantity. Relevant in numerous applications (e.g. wireless communications).
- The bandwidth determines the minimum sampling rate (Nyquist rate) that is necessary to reconstruct a bandlimited signal from its samples (Shannon sampling series).

$$B(f) \to r_{\text{min}} = B(f)/\pi \to \{f(k/r_{\text{min}})\}_{k\in\mathbb{Z}}$$

- The sequence of samples $\{f(k\pi/B(f))\}_{k\in\mathbb{Z}}$, taken at Nyquist rate, can therefore be seen as a minimum representation of the signal f (no loss of information).

Can we determine the actual bandwidth $B(f)$ of a bandlimited signal f, i.e., the smallest number σ such that f is bandlimited with bandwidth σ, on a digital computer?

Turing Machine I

Turing Machine:
Abstract device that manipulates symbols on a strip of tape according to certain rules.

- Turing machines are an idealized computing model.
- No limitations on computing time or memory, no computation errors.
- Although the concept is very simple, Turing machines are capable of simulating any given algorithm.

Turing machines are suited to study the limitations in performance of a digital computer:

> Anything that is not Turing computable cannot be computed on a real digital computer, regardless how powerful it may be.

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proceedings of the London Mathematical Society*, vol. s2-42, no. 1, pp. 230–265, Nov. 1936

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem. A correction," *Proceedings of the London Mathematical Society*, vol. s2-43, no. 1, pp. 544–546, Jan. 1937

Turing Machine II

- There exist problems that cannot be solved on a digital computer.
- For example, computation of Fourier transform or spectral factorization for certain signals.
- The computer cannot produce, for any desired error ϵ, a result that ϵ-close to the true value.
  $\rightarrow$ the approximation error cannot be controlled.

Questions

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1. Is $B(f)$ computable?
2. Can we compute a lower bound for $B(f)$?
3. Can we compute an upper bound for $B(f)$?

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In terms of sampling rate, those questions read:

1. Is the optimal, i.e. minimum required sampling rate computable?
2. Can we compute a lower bound for the minimum required sampling rate?
3. Can we compute an upper bound for the minimum required sampling rate?

Overview of the Results

Actual bandwidth $B(f)$ computable?	Determine if σ is a lower bound for $B(f)$?	Determine if σ is an upper bound for $B(f)$?
x No	✓ Yes	x No

Notation

- $L^p(\Omega)$, $1 \leqslant p < \infty$: space of all measurable, pth-power Lebesgue integrable functions on Ω
  Norm: $\|f\|_p = \left(\int_\Omega |f(t)|^p \, dt\right)^{1/p}$
- $L^\infty(\Omega)$: space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite

Bandlimited Functions

Definition (Bernstein Space)

Let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leqslant C(\epsilon) \exp\big((\sigma + \epsilon)|z|\big)$ for all $z \in \mathbb{C}$.

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line.

- A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ.
- $B(f)$: actual bandwidth of the function $B(f) = \min\{\sigma \in \mathbb{R} \colon f \in \mathcal{B}_\sigma\}$
- We have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r$ for all $1 \leqslant p \leqslant r \leqslant \infty$.
- $\mathcal{B}_\sigma^2$: space of bandlimited functions with finite energy.

Actual Bandwidth for $\mathcal{B}_\pi^2$

For $f \in \mathcal{B}_\pi^2$ we have a simple characterization of the actual bandwidth.

- $B(f)$ is the smallest number $\sigma > 0$ such that

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \, d\omega.$$

- $B(f)$ is the smallest $\sigma > 0$ such that

$$f(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \hat{f}(\omega) \, e^{i\omega t} \, d\omega$$

for all $t \in \mathbb{R}$.

Computable Sequences of Rationals

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

- A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.

Computable Real Numbers

First example of an effective approximation

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|x - r_n| \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathbb{R}_c$: set of computable real numbers
- $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable.
- Commonly used constants like e and π are computable.

Computable Bandlimited Functions I

We call a function f elementary computable if there exists a natural number L and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

- Every elementary computable function is Turing computable.
- For every elementary computable function f, the norm $\|f\|_{\mathcal{B}_\pi^p}$ is computable.

Computable Bandlimited Functions II

A function in $f \in \mathcal{B}_\pi^p$ is called computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

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for all $n \geqslant \xi(M)$.

- $\mathcal{CB}_\pi^p$: set of all functions that are computable in $\mathcal{B}_\pi^p$.

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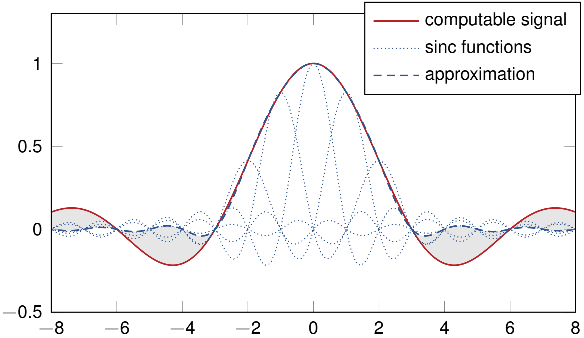

Computability of the Actual Bandwidth

Question:
Does there exist an algorithm that, for every computable signal $f \in \mathcal{CB}_\pi^1$ (or $f \in \mathcal{CB}_\pi^2$), is able to compute $B(f)$?

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- Necessary condition: B maps computable functions into computable numbers ($B\colon \mathcal{CB}^1_\pi \to \mathbb{R}_c$).

Weaker question:

Do we have $B(f) \in \mathbb{R}_c$ for all $f \in \mathcal{CB}^1_\pi$ (or $f \in \mathcal{CB}^2_\pi$)?

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Theorem

There exists a signal $f_1 \in \mathcal{CB}_\pi^1$ (and $\mathcal{CB}_\pi^2$) such that $B(f_1) \notin \mathbb{R}_c$, i.e., $B(f_1)$ is not Turing computable.

- The actual bandwidth is not always computable.

Semi Decidability I

- There are problematic signals f, for which we cannot compute $B(f)$.
- Can we at least algorithmically determine whether, for a given signal f, we can compute $B(f)$, or not?
- Would be helpful to avoid problematic signals, e.g., in automated computer aided design (CAD).

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We call a set $\mathcal{M} \subset \mathcal{CB}_\pi^1$ semi-decidable if there exists a Turing machine

$$\text{TM}\colon \mathcal{CB}_\pi^1 \to \{\text{TM stops, TM runs forever}\}$$

that, given an input $f \in \mathcal{CB}_\pi^1$, stops if and only if $f \in \mathcal{M}$.

Semi Decidability II

Set of all signals in $\mathcal{CB}^1_\pi$ for which $B(f)$ can be computed algorithmically:

$$\mathcal{C}^1_{\mathsf{BW}} = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) \in \mathbb{R}_c \right\}$$

Set of all signals in $\mathcal{CB}^1_\pi$, for which $B(f)$ cannot be computed algorithmically:

$$\mathcal{NC}^1_{\mathsf{BW}} = \mathcal{CB}^1_\pi \setminus \mathcal{C}^1_{\mathsf{BW}} = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) \notin \mathbb{R}_c \right\}$$

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Set of all signals in $\mathcal{CB}_\pi^1$ for which $\mathrm{B}(f)$ can be computed algorithmically:

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Can we determine algorithmically whether $f \in \mathcal{NC}_{\mathrm{BW}}^1$?

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Set of all signals in $\mathcal{CB}^1_\pi$, for which $\mathrm{B}(f)$ cannot be computed algorithmically:

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Can we determine algorithmically whether $f \in \mathcal{NC}^1_{\mathrm{BW}}$?

Theorem

Neither $\mathcal{C}^1_{BW}$ nor $\mathcal{NC}^1_{BW}$ is semi-decidable.

Approximate Bandwidth I

For $\sigma > 0$, we want to algorithmically determine if $B(f) > \sigma$.

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Theorem

For all $\sigma \in (0, \pi) \cap \mathbb{R}_c$ the set

$$\mathcal{C}^1_{>}(\sigma) = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) > \sigma \right\}$$

is semi-decidable.

- There exists an algorithm that stops if and only if $B(f) > \sigma$.

Approximate Bandwidth I

For $\sigma > 0$, we want to algorithmically determine if $B(f) > \sigma$.

Theorem

For all $\sigma \in (0, \pi) \cap \mathbb{R}_c$ the set

$$\mathcal{C}^1_>(\sigma) = \{f \in \mathcal{CB}^1_\pi : B(f) > \sigma\}$$

is semi-decidable.

- There exists an algorithm that stops if and only if $B(f) > \sigma$.
- Does not allow us to determine an effective upper bound for $B(f)$, because this Turing machine does not stop if $B(f) \leqslant \sigma$.

Approximate Bandwidth II

Theorem

For all $\sigma \in (0, \pi) \cap \mathbb{R}_c$ *the set*

$$\mathcal{C}^1_{\leqslant}(\sigma) = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) \leqslant \sigma \right\}$$

is not semi-decidable.

Approximate Bandwidth II

Theorem

For all $\sigma \in (0,\pi) \cap \mathbb{R}_c$ *the set*

$$\mathcal{C}^1_{\leqslant}(\sigma) = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) \leqslant \sigma \right\}$$

is not semi-decidable.

Consequence:

- For a given $\sigma \in (0,\pi) \cap \mathbb{R}_c$, we cannot determine algorithmically for all $f \in \mathcal{CB}^1_\pi$ whether f is uniquely determined by the samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$.

Conclusions

- We studied if it is possible to algorithmically determine the actual bandwidth of a bandlimited signal.
- We proved that this is not possible in general.
- The minimal sampling rate cannot be determined algorithmically in general.

Thank you!