EXTENDED CYCLIC COORDINATE DESCENT FOR ROBUST ROW-SPARSE SIGNAL RECONSTRUCTION IN THE PRESENCE OF OUTLIERS

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Content

Overview

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Introduction
Sparse Signal Reconstruction (1)

- A sparse signal can be reliably recovered from a reduced set of measurements, under certain conditions, e.g., RIP.

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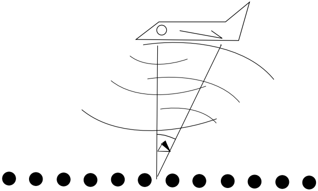
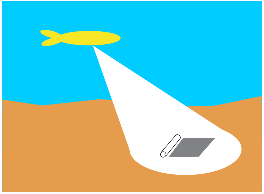
- A sparse signal can carry useful information, such as directions of sources.
- Applications



(a) Target tracking

(b) Through-the-wall radar 3D reconstruction



(c) Sonar imaging

Introduction
Sparse Signal Reconstruction (2)

Data model:

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{e}(t), \quad t = 1, 2, \ldots, T$$

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- $\mathbf{y}(t) \in \mathbb{C}^N$: Measurement data
- $\mathbf{A} \in \mathbb{C}^{N \times P}$: Predictor dictionary, $N < P$
- $\mathbf{s}(t) \in \mathbb{C}^P$: Signal vector
- $\mathbf{e}(t) \in \mathbb{C}^N$: Gaussian noise

Introduction
Sparse Signal Reconstruction (3)

Data model in matrix form:

$$\mathbf{Y} = \mathbf{A}\mathbf{S} + \mathbf{E}$$

where $\mathbf{Y} = [\mathbf{y}(1), \mathbf{y}(2), \cdots, \mathbf{y}(T)]$, $\mathbf{S} = [\mathbf{s}(1), \mathbf{s}(2), \cdots, \mathbf{s}(T)]$, $\mathbf{E} = [\mathbf{e}(1), \mathbf{e}(2), \cdots, \mathbf{e}(T)]$

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Sparse signal reconstruction using LASSO:

$$\min_{\mathbf{S}} \quad \frac{1}{2}\|\mathbf{Y} - \mathbf{A}\mathbf{S}\|_F^2 + \lambda\|\mathbf{S}\|_{2,1}$$

Introduction
Non-Gaussian Heavy-Tailed Noise (1)

- Data model without outliers

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{e}(t)$$

$\mathbf{e}(t) \in \mathbb{C}^N$: Gaussian noise

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$\mathbf{q}(t) \in \mathbb{R}^N$: Weight vector

$$\begin{cases} q_i(t) \gg 1 & : \text{Outlier} \\ q_i(t) = 1 & : \text{Non-outlier} \end{cases}$$

where $q_i(t)$ is the i-th entry of $\mathbf{q}(t)$

Introduction
Non-Gaussian Heavy-Tailed Noise (2)

- Data model with outliers in matrix form:

$$\mathbf{Y} = \mathbf{AS} + \mathbf{Q} \circ \mathbf{E}$$

where $\mathbf{Q} = [\mathbf{q}(1), \mathbf{q}(2), \cdots, \mathbf{q}(T)]$

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- The problem to be addressed is as follows:

  With known **Y** and **A**, recover the row-sparse matrix **S.**

Overview

Proposed Approach
Proposed Minimization Problem (1)

$$\min_{\mathbf{W},\mathbf{S}} \quad \frac{1}{2}\|\mathbf{W} \circ (\mathbf{Y} - \mathbf{AS})\|_F^2 + \lambda_1 \|\mathrm{vec}(\mathbf{W}_{-1})\|_1 + \lambda_2 \|\mathbf{S}\|_{2,1}$$

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Proposed Minimization Problem (1)

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- The first term: **W** is used to down-weight the the outliers while keeping the remaining observations unchanged

$$0 < W_{ij} < 1 \text{ for outliers, } W_{ij} = 1 \text{ for non-outliers}$$

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- The second term: Impose sparsity of $\mathbf{W}_{-1} = \mathbf{W} - \mathbf{1}_{N \times T}$
- The third term: Impose row-sparsity of **S**

Proposed Approach
Proposed Minimization Problem (2)

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$$\boxed{\min_{\mathbf{W},\mathbf{S}} \quad \frac{1}{2}\|\mathbf{W} \circ (\mathbf{Y} - \mathbf{AS})\|_F^2 + \lambda_1 \|\mathrm{vec}(\mathbf{W}_{-1})\|_1 + \lambda_2 \|\mathbf{S}\|_{2,1}}$$

- LASSO-type problem, cyclic coordinate descent (CCD)

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- LASSO-type problem, cyclic coordinate descent (CCD)
- Biconvex with respect to **W** and **S**

Proposed Approach
Remarks (1)

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$$\min_{\mathbf{W},\mathbf{S}} \frac{1}{2}\|\mathbf{W} \circ (\mathbf{Y} - \mathbf{AS})\|_F^2 + \lambda_1 \|\mathrm{vec}(\mathbf{W}_{-1})\|_1 + \lambda_2 \|\mathbf{S}\|_{2,1}$$

- Comparison with the one using Huber's loss function

$$\min_{\mathbf{S}} \|\rho_{\mathrm{H}}(\mathbf{Y} - \mathbf{AS})\|_F^2 + \mu\|\mathbf{S}\|_{2,1}$$

where

$$\rho_{\mathrm{H}}(x) = \begin{cases} |x|^2, & \text{for } |x| \leq c \ \text{(i.e., non-outliers)} \\ 2c|x| - c^2, & \text{for } |x| > c \ \text{(i.e., outliers)} \end{cases}$$

Proposed Approach
Remarks (2)

$$\min_{\mathbf{W},\mathbf{S}} \quad \frac{1}{2}\|\mathbf{W} \circ (\mathbf{Y} - \mathbf{AS})\|_F^2 + \lambda_1 \|\mathrm{vec}(\mathbf{W}_{-1})\|_1 + \lambda_2 \|\mathbf{S}\|_{2,1}$$

- Comparison with the original problem without outliers

$$\min_{\mathbf{S}} \quad \frac{1}{2}\|\mathbf{Y} - \mathbf{AS}\|_F^2 + \lambda \|\mathbf{S}\|_{2,1}$$

Proposed Approach
Extended Cyclic Coordinate Descent Algorithm

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- Step 1: Update W_{ij} by fixing $\mathbf{S}$ and W_{nt} ($n = 1, \ldots, N$ and $n \neq i$, $t = 1, \ldots, T$ and $t \neq j$)

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- Step 2: Update S_{ij} by fixing $\mathbf{W}$ and S_{pt} ($p = 1, \ldots, P$ and $p \neq i$, $t = 1, \ldots, T$ and $t \neq j$)

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Detailed derivations of the whole algorithm can be found in our paper.

Overview

Application to Source Localization
Relative Frequency (RF) Comparison

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Relative Frequency (RF) Comparison

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Setups:

- ULA of $N = 8$ sensors
- $K = 3$ signals with directions: $\{-20^\circ, 0^\circ, 15^\circ\}$
- Over-complete dictionary $\mathbf{A}$ contains $P = 101$ columns, covering from -50° to 50°
- $T = 64$ snapshots
- $\text{SNR} = 0$ dB

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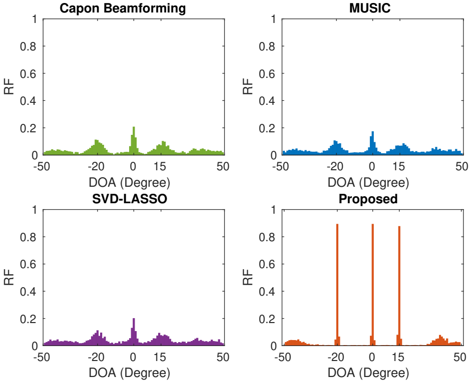

Application to Source Localization
Probability of Exact Recovery (PER) Comparison

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Setups:

- ULA of $N = 5$ sensors
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Probability of Exact Recovery (PER) Comparison

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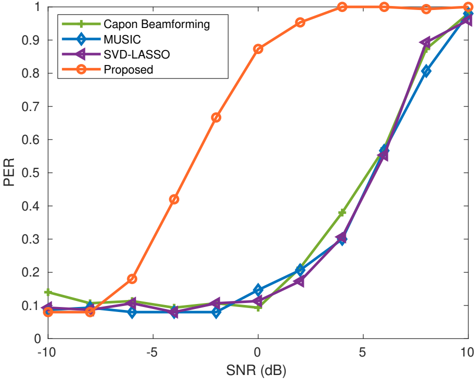

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Summary and Outlook

Summary:

- This paper studied the problem of row-sparse signal recovery for complex-valued data contaminated with outliers.
- Using a weight matrix, we formulated the problem in the form of LASSO-type.
- We extended CCD algorithm to solve the resulting problem.
- Simulation results showed that the proposed algorithm is robust against outliers even in the low SNR regime.

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Outlook:

- The sparsity structure of the weight matrix will be further studied.
- Some state-of-the-art techniques, such as SPARROW[1], will be studied in the framework of row-sparse signal reconstruction.

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Thank you for your attention!

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