Tensor Decomposition-based Beamspace ESPRIT Algorithm for Multidimensional Harmonic Retrieval

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Outline

1. Problem Formulation
2. Proposed Method
3. Numerical Results
4. Conclusions

Multidimensional (R-D) Harmonic Retrieval (HR)

For the kth snapshot, the **element-space** tensor $\boldsymbol{\mathcal{X}}_k$ has entries of the form:[1]

$$x_{m_1,\cdots,m_R,k} = \sum_{l=1}^{L} \gamma_{l,k} \prod_{r=1}^{R} e^{j m_r \omega_{r,l}}, \tag{1}$$

where $m_r = 0, 1, \cdots, M_r - 1$*. Here,* M_r*,* R *and* L *denote the number of sensors for the* r*th dimension, the number of dimensions and the number of* R*-D frequencies, respectively,* $\gamma_{l,k}$ *represents the complex amplitude of the* l*th frequency at the* k*th snapshot, while* $\omega_{r,l} \in (-\pi, \pi)$ *is the frequency in the* r*th dimension of the* l*th source.*

The tensor $\boldsymbol{\mathcal{X}}_k$ can be expressed as

$$\boldsymbol{\mathcal{X}}_k = \sum_{l=1}^{L} \gamma_{l,k}\, \mathbf{a}_{1,l} \circ \mathbf{a}_{2,l} \circ \cdots \circ \mathbf{a}_{R,l} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R} \tag{2}$$

where $\mathbf{a}_{r,l} = [1 \;\; e^{j\omega_{r,l}} \;\; \cdots \;\; e^{j(M_r-1)\omega_{r,l}}]^{\mathsf{T}}$ *and* $\circ$ *is the vector outer product.*

[1] For notational compactness, noise is omitted in the equations

From Element-space to Beamspace

Beamspace processing is an efficient and commonly used approach in HR. The measurements are obtained by *linearly transforming* the sensing data, thereby achieving a compromise between estimation accuracy and system complexity.

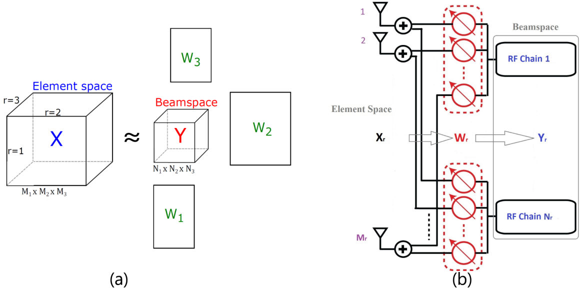

Figure: Illustration of measurements in beamspace (a) general case. (b) MIMO example with hybrid combining (hardware constraints).

Beamspace Model

For beamspace measurements, after the r-mode product of $\boldsymbol{\mathcal{X}}_k$ with linear transformation matrix $\mathbf{W}_r$, the model (2) is modified to

$$\boldsymbol{\mathcal{Y}}_k = \sum_{l=1}^{L} \gamma_{l,k}\, \mathbf{b}_{1,l} \circ \mathbf{b}_{2,l} \circ \cdots \circ \mathbf{b}_{R,l}, \tag{3}$$

where the beamspace array manifold is defined as

$$\mathbf{B}_r = [\mathbf{b}_{r,1}\ \mathbf{b}_{r,2}\ \cdots\ \mathbf{b}_{r,L}] = \mathbf{W}_r^{\mathsf{H}} \mathbf{A}_r \in \mathbb{C}^{N_r \times L}. \tag{4}$$

Here $\mathbf{W}_r^{\mathsf{H}} = [\mathbf{w}_{r,1}\ \ \mathbf{w}_{r,2}\ \ \cdots\ \ \mathbf{w}_{r,M_r}] \in \mathbb{C}^{N_r \times M_r}$, $\mathbf{W}_r^{\mathsf{H}} \mathbf{W}_r = \mathbf{I}_{N_r}$ is required to maintain whiteness in the beamspace output, and the element-space array manifold

$$\mathbf{A}_r = [\mathbf{a}_{r,1}\ \ \mathbf{a}_{r,2}\ \ \cdots\ \ \mathbf{a}_{r,L}] \in \mathbb{C}^{M_r \times L}. \tag{5}$$

$\Rightarrow$ *The transformation matrix $\mathbf{W}_r$ and number N_r should be chosen properly to cover the sector of source locations and most of the signal energy.*

Objective

Our objective is to estimate $\omega_{r,l}$, for $r = 1, \cdots, R$ and $l = 1, \cdots, L$, from noisy measurements $\tilde{\boldsymbol{\mathcal{Y}}}_k$.

- ☐ Computationally Efficient $\Rightarrow$ Subspace-based Search-free method
- ☐ Automatic Association $\Rightarrow$ Joint parameter Estimation and Association (For path l, what is $\omega_{1,l}, \omega_{2,l}, \cdots, \omega_{R,l}$?)

A number of HR techniques are available in the literature (maximum likelihood, subspace, compressed sensing, ...)

Estimation of signal parameters via rotational invariance techniques (ESPRIT) and its variants have become one of the popular search-free signal **subspace-based** *parameter estimation methods.*

$\Rightarrow$ **Beamspace tensor-ESPRIT**

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Idea 1: Multidimensional Parameter Association

In the CP decomposition, a tensor is decomposed into a sum of rank-one component tensors,

$$\tilde{\mathcal{Y}}_k = \sum_{l=1}^{L} \lambda_l \, \mathbf{u}_{1,l} \circ \mathbf{u}_{2,l} \circ \cdots \circ \mathbf{u}_{R,l}. \quad (6)$$

Both association and noise reduction are achieved simultaneously.

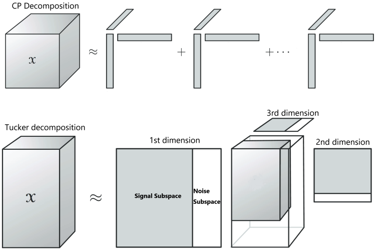


Figure: Illustration of 3-D CP and Tucker tensor decomposition.

Idea 2: Shift Invariant Property

ESPRIT algorithms utilize the shift invariant property:

$$\mathbf{J}_r^{(1)}\mathbf{A}_r = \mathbf{J}_r^{(2)}\mathbf{A}_r\mathbf{\Phi}_r, \tag{7}$$

where $\mathbf{\Phi}_r$ contains the frequencies of all sources in rth dimension,

$$\mathbf{\Phi}_r = \mathrm{diag}\begin{bmatrix} e^{-j\omega_{r,1}} & e^{-j\omega_{r,2}} & \cdots & e^{-j\omega_{r,L}} \end{bmatrix}, \tag{8}$$

$\mathbf{J}_r^{(1)} = \begin{bmatrix} \mathbf{I}_{N_r-1} & \mathbf{0}_{(N_r-1)\times 1} \end{bmatrix}$ and $\mathbf{J}_r^{(2)} = \begin{bmatrix} \mathbf{0}_{(N_r-1)\times 1} & \mathbf{I}_{N_r-1} \end{bmatrix}$ are selection matrices.

In beamspace, the row transformation $\mathbf{W}_r^{\mathsf{H}}$ alters the transitional invariance structure in the array manifold, and consequently

$$\mathbf{J}_r^{(1)}\mathbf{B}_r \neq \mathbf{J}_r^{(2)}\mathbf{B}_r\mathbf{\Phi}_r. \tag{9}$$

However, the shift invariance structure can be restored, if $\mathbf{W}_r$ has a **similar structure**.

Proposed Method

Suppose we are able to find a non-singular $N_r \times N_r$ matrix $\mathbf{F}_r$ that satisfies

$$\mathbf{J}_r^{(1)}\mathbf{W}_r = \mathbf{J}_r^{(2)}\mathbf{W}_r\mathbf{F}_r. \tag{10}$$

and

$$\mathbf{Q}_r = \mathbf{I}_{N_r} - \mathbf{w}_{r,M_r}\mathbf{w}_{r,M_r}^{\mathsf{H}} - \left(\mathbf{F}_r^{\mathsf{H}}\mathbf{w}_{r,1}\right)\left(\mathbf{F}_r^{\mathsf{H}}\mathbf{w}_{r,1}\right)^{\mathsf{H}}. \tag{11}$$

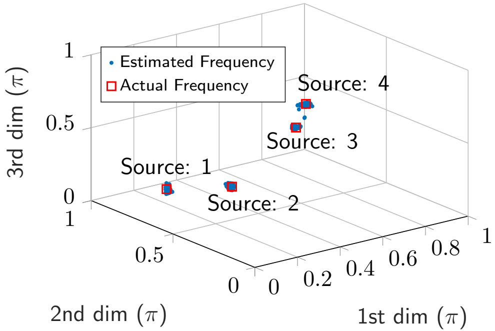
Then

$$\mathbf{Q}_r\mathbf{F}_r^{\mathsf{H}}\mathbf{U}_r = \mathbf{Q}_r\mathbf{U}_r\mathbf{\Gamma}_r \tag{12}$$

where $\mathbf{U}_r = [\mathbf{u}_{r,1} \quad \mathbf{u}_{r,2} \quad \cdots \quad \mathbf{u}_{r,L}]$, its columns span the signal subspace, $\mathbf{\Gamma}_r = \mathbf{D}_r\mathbf{\Phi}_r^{\mathsf{H}}\mathbf{D}_r^{-1} \in \mathbb{C}^{L\times L}$ and $\mathbf{D}_r \in \mathbb{C}^{L\times L}$ is a non-singular matrix.

1. CP decomposition on $\tilde{\mathcal{Y}}_k \Rightarrow \mathbf{U}_r, r = 1, 2, \cdots, R$.
2. Estimate $\mathbf{F}_r$ from (10), construct $\mathbf{Q}_r$ by (11), estimate $\mathbf{\Gamma}_r$ from (12)
3. Frequency $\omega_{r,l}$ is obtained from the lth eigenvalue of $\mathbf{\Gamma}_r$.

Test 1: Parameter estimation for partially distinct frequencies.



4 sources with partially distinct frequencies.

In element space,

$M_1 = M_2 = M_3 = 8.$

In beamspace,

$N_1 = N_2 = N_3 = 6.$

SNR=20 dB, and the number of measurements is K=10.

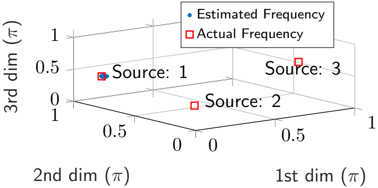
The 3-D HR frequencies are:

$r = 1 : (\boxed{0.2\pi, 0.2\pi}, 0.6\pi, 0.8\pi)$

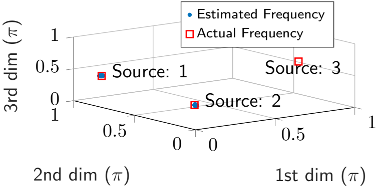
$r = 2 : (0.9\pi, \boxed{0.4\pi, 0.4\pi}, 0.6\pi)$

$r = 3 : (0.1\pi, 0.2\pi, \boxed{0.8\pi, 0.8\pi}).$

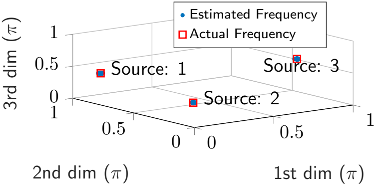
Test 2: Inaccurate source number information (3 sources).



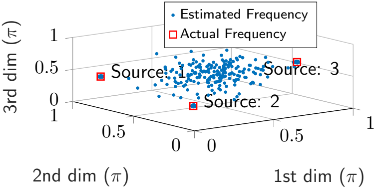
(a) Assumed number of sources $\hat{L} = 1$.



(b) Assumed number of sources $\hat{L} = 2$.



(c) Assumed number of sources $\hat{L} = 3$.



(d) Assumed number of sources $\hat{L} = 4$.

Conclusions

A beamspace R-D tensor-ESPRIT algorithm is developed for multidimensional harmonic retrieval.

Source parameter estimation and association are achieved simultaneously.

Furthermore, the effect of *errors in the estimated number of sources* is investigated, as well as the applicability for *sources with partially distinct frequencies* is demonstrated.