TECHNISCHE UNIVERSITÄT WIEN

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LEARNING SIGNED GRAPHS FROM DATA

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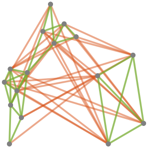
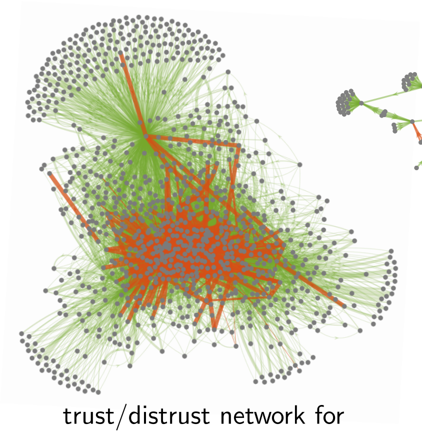
Motivation

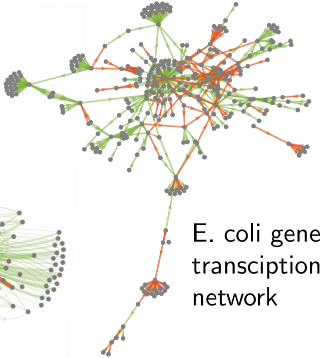
Signed graphs have positive and negative edge weights to model similarity/dissimilarity, like/dislike, trust/distrust, friend/foe, activation/inhibition



friend-foe network for 16 Guinea highland tribes



trust/distrust network for 1000 users on Bitcoin Alpha



E. coli gene transciption network

Problem Statement

Given: N objects, M measurements $\rightarrow N \times M$ data matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_N^T \end{pmatrix} = \left(\tilde{\mathbf{x}}_1 \ldots \tilde{\mathbf{x}}_M\right) \qquad \begin{cases} N \text{ measurement vectors } \mathbf{x}_i \text{ (rows)} \\ M \text{ graph signals } \tilde{\mathbf{x}}_m \text{ (columns)} \end{cases}$$

Signed graph learning ➡ find edge weight matrix $\mathbf{W} \in \mathbb{R}^{N \times N}$

$$\begin{cases} W_{ij} = 0, & \text{no edge between nodes } i \text{ and } j \\ W_{ij} > 0, & \text{similarity edge between nodes } i \text{ and } j \\ W_{ij} < 0, & \text{dissimilarity edge between nodes } i \text{ and } j \end{cases}$$

Rationale:

- graph signals $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_M$ shall be smooth on the graph
- $+/-$ edge between nodes i and j if $\mathbf{x}_i$ and $\mathbf{x}_j$ are similar/dissimilar

NB: unsigned graph learning ($W_{ij} \geq 0$) has received a lot of attention!

Smoothness/(Dis)similarity Metrics

Signed Laplacian form:

$$\mathbf{x}^T\mathbf{L}\mathbf{x} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}|W_{ij}| \cdot \big(x_i - \mathsf{sign}(W_{ij})x_j\big)^2$$

with signed Laplacian

$$\mathbf{L} = \mathsf{Diag}\{d_1, \ldots, d_N\} - \mathbf{W}, \qquad d_i = \sum_{j=1}^{N}|W_{ij}|$$

Signed total variation:

$$\|\mathbf{x}\|_{\mathsf{TV}} = \sum_{i=1}^{N}\sum_{j=1}^{N}|W_{ij}| \cdot |x_i - \mathsf{sign}(W_{ij})x_j|$$

Both metrics are small if $x_i \approx x_j$ for $W_{ij} > 0$ and $x_i \approx -x_j$ for $W_{ij} < 0$

Optimization Problem

Unified formulation of cost function:

$$f_p(\mathbf{W}) = \sum_{i=1}^{N} \sum_{j=1}^{N} |W_{ij}| \cdot \|\mathbf{x}_i - \mathsf{sign}(W_{ij})\mathbf{x}_j\|_p^p$$

Optimization problem:

$$\min_{\mathbf{W} \in \mathcal{W}} \; f_p(\mathbf{W}) + \frac{\mu}{2}\|\mathbf{W}\|_2^2$$

$$\mathcal{W} = \left\{ \mathbf{W} : \begin{array}{ll} 0 \leq |W_{ij}| \leq C_{ij}, & \ldots \text{ prior knowledge} \\ a_i \leq \sum_{j=1}^{N} |W_{ij}| \leq b_i, & \ldots \text{ control node degrees} \\ \sum_{j=1}^{N} |W_{ij}| = 2N & \ldots \text{ weight normalization} \end{array} \right\}$$

This problem is not convex ...

Solution

Sign-magnitude representation: $W_{ij} = |W_{ij}| \operatorname{sign}(S_{ij}) = \widetilde{W}_{ij} S_{ij}$

$$\Rightarrow \min_{\substack{\widetilde{W}_{ij} \in \mathcal{W} \\ S_{ij} \in \{-1,1\}}} \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{W}_{ij} \|\mathbf{x}_i - S_{ij}\mathbf{x}_j\|_p^p + \frac{\mu}{2} \widetilde{W}_{ij}^2$$

Sign optimization has explicit solution:

$$\min_{S_{ij} \in \{-1,1\}} \|\mathbf{x}_i - S_{ij}\mathbf{x}_j\|_p^p = D_{ij}^{(p)} \triangleq \min \Big\{ \underbrace{\|\mathbf{x}_i - \mathbf{x}_j\|_p^p}_{S_{ij}=1}, \underbrace{\|\mathbf{x}_i + \mathbf{x}_j\|_p^p}_{S_{ij}=-1} \Big\}$$

Magnitude optimization amounts to ordinary unsigned graph learning:

$$\min_{\widetilde{\mathbf{W}} \in \mathcal{W}} \operatorname{tr}\{\widetilde{\mathbf{W}} \mathbf{D}^{(p)}\} + \frac{\mu}{2} \|\widetilde{\mathbf{W}}\|_2^2$$

. . . quadratic problem, can be solved efficiently e.g. using ADMM

Illustrative Example

$N = 3$, $\mathbf{x}^T = (1, -2, 4)$, $p = 1$, no weight and degree constraints

$|x_1 - x_2| = 3$

$|x_1 + x_2| = 1$

$S_{12} = -1,\ D_{12}^{(1)} = 1$

$|x_1 - x_3| = 3$

$|x_1 + x_3| = 5$

$S_{13} = 1,\ D_{13}^{(1)} = 3$

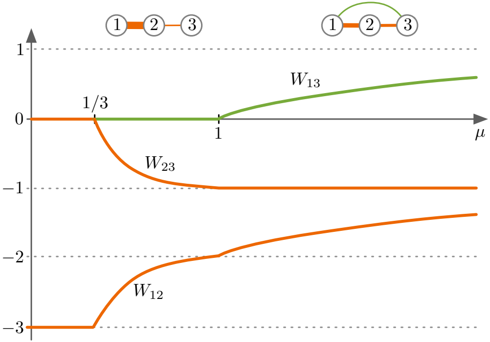
$|x_2 - x_3| = 6$

$|x_2 + x_3| = 2$

$S_{23} = -1,\ D_{23}^{(1)} = 2$

Waterfilling solution:

$$W_{ij} = \frac{S_{ij}}{\mu} \max\left\{0, \omega - D_{ij}^{(1)}\right\}$$



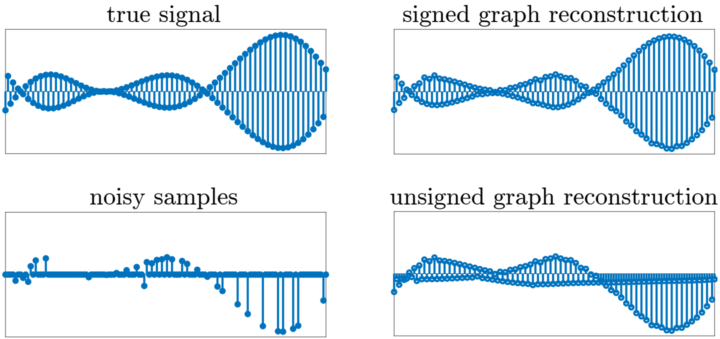
Application 1: Signal Reconstruction

Given: K noisy samples $y_i = s_i + z_i$, $i \in \mathcal{S}$

Goal: reconstruct bandpass signal $\mathbf{s}$ of length $N = 128$

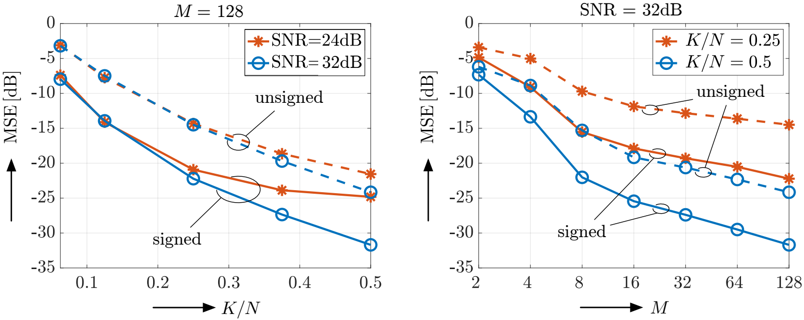
Laplacian regression: $\mathbf{s}^{\text{opt}} = \arg\min_{\mathbf{s}} \mathbf{s}^T \mathbf{L} \mathbf{s} + \nu \sum_{i \in \mathcal{S}} (y_i - s_i)^2$

Signed/unsigned graph learned from M examples $\tilde{\mathbf{x}}_m = \mathbf{s}_m + \mathbf{z}_m$ ($p = 2$)



Reconstruction Performance

Reconstruction MSE obtained via 1000 Monte Carlo trials



- signed outperforms unsigned by typically 7 dB
- unsigned requires twice as many samples as signed

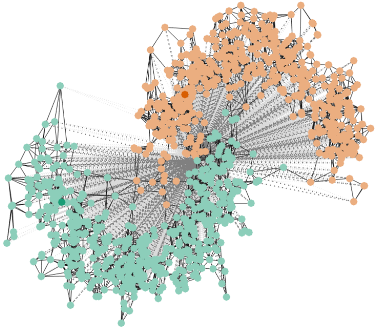
Application 2: Clustering

Given: $N = 1000$ points $\mathbf{x}_i$ of the two-moon model ($M = 2$)

Goal: Find cluster labels $\{-1, 1\}^N$

Total variation clustering: $\mathbf{y}^{\mathsf{opt}} = \underset{\mathbf{y}\in[-1,1]^N}{\arg\min} \|\mathbf{y}\|_{\mathsf{TV}}$

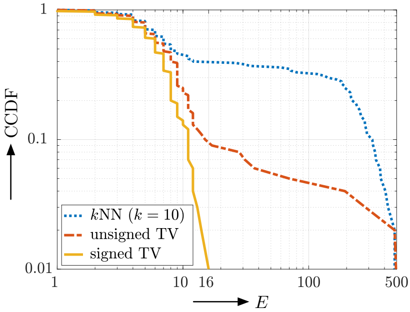
Signed/unsigned graph learned from $\mathbf{x}_i$ with $p = 1$



signed clustering unsigned clustering

Clustering Performance

Misclassification rate E obtained from 100 Monte Carlo runs



- signed clustering: average $E = 6.42 \pm 3.01$, worst-case $E = 16$
- unsigned clustering: average $E = 22.64 \pm 74.92$, worst-case $E = 500$
- kNN clustering: average $E = 93.75 \pm 133.4$, worst-case $E = 500$

Conclusion

- Signed graphs offer increased modeling flexibility
- Proposed learning strategy for signed graphs
  - based on notions of (dis)similarity and smoothness
  - non-convex problem, can be reduced to unsigned learning
  - efficient implementation
- Application examples
  - signal sampling and reconstruction
  - data clustering
- at same graph sparsity, signed outperforms unsigned
- Future work: distinct penalty and constraints for $+/-$ edges

THANKS FOR WATCHING!

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