Can every analog system be simulated on a digital computer?

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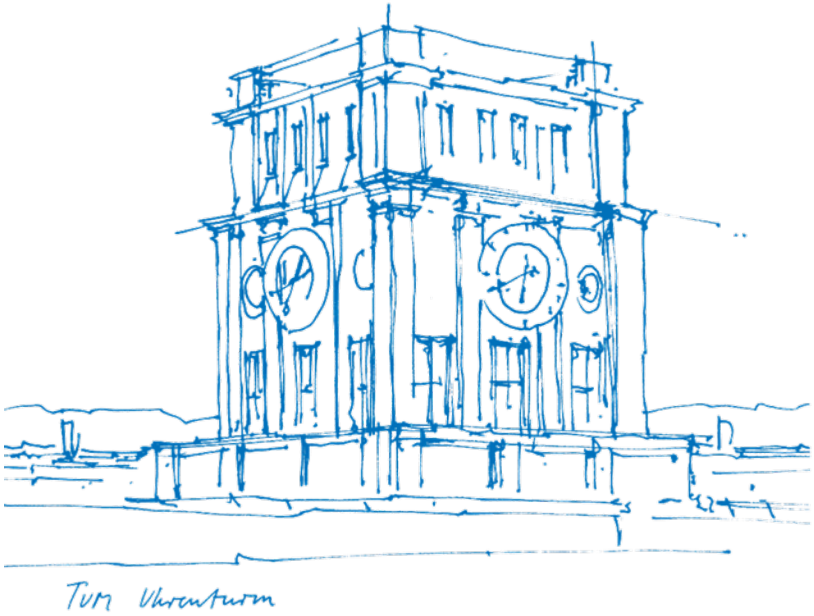
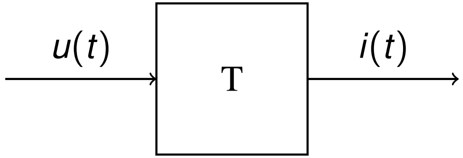


Motivation – Simulation of Continuous LTI Systems

- Consider a linear *time-invariant (LTI) system* T mapping an input signal $u(t)$ onto an output signal $i(t)$.



- Input and output are described by *continuous* variables $u(t)$ and $i(t)$, respectively, which live in an *uncountable set* (e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^N$). Often, these quantities are currents, voltages, potentials, etc.
- The input and output variables often depend on a *continuous* parameters t in *uncountable set* like. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^N$, etc. These parameters t often describes „time“ or „space“ (position).
- Assume there is a mathematical model for T describing the relation between input and output: $i(t) = (\mathrm{T}u)(t)$

Question:

Assume u is an arbitrary *admissible input* for T. Is it possible to calculate $i(t) = (\mathrm{T}u)(t)$ on a *digital* computer?

Problem: Digital computers can exactly solve only finite discrete problems.

Outline

1. Review of computability theory
   - computable numbers, computable functions, Turing machines, etc.

2. The most simple LTI system with non-computable output – Ideal capacitor

3. Classes of input signals with a computable output

4. Summary and outlook

Computability Analysis

Computable Rational Numbers

Definition: A sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers is said to be computable if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ and such that

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}\,, \qquad n \in \mathbb{N}\,.$$

A recursive function $a : \mathbb{N} \to \mathbb{N}$ is a mapping that is build form elementary computable functions and recursion and can be calculated on a *Turing machine*.

Turing machine

- can simulate any given algorithm and therewith provide a simple but very powerful model of computation.
- is a theoretical model describing the fundamental limits of any realizable digital computer.
- Most powerful programming languages are called Turing-complete (such as C, C++, Java, etc.).

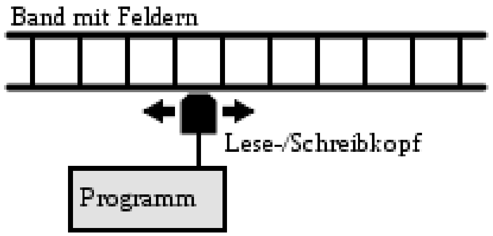


Figure taken from *Wikipedia*

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. s2-42, no. 1, 1937.

Computable Real Numbers

- ▷ Any real number $x \in \mathbb{R}$ is the limit of a sequence of rational numbers.
- ▷ For $x \in \mathbb{R}$ to be computable, the convergence has to be effective.

Definition (Computable number): A real number $x \in \mathbb{R}$ is said to be *computable* if there exists a computable sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers which *converges effectively* to x,
i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$

$$|x - r_n| \leq 2^{-N} \qquad \text{whenever} \quad n \geq e(N)\,.$$

$\Rightarrow$ $x \in \mathbb{R}$ is computable if a Turing machine can approximate it with exponentially vanishing error.

- $\mathbb{R}_c$ stand for the set of all *computable real numbers*.
- $\mathbb{C}_c = \{x + iy : x, y \in \mathbb{R}_c\}$ stands for the set of all *computable complex numbers*.
- Note that the set of computable numbers $\mathbb{R}_c \subsetneq \mathbb{R}$ is only countable.

Computable Functions

Definition: A function $f : \mathbb{T} \to \mathbb{R}$ on an interval $\mathbb{T} \subset \mathbb{R}$ is said to be computable if

(a) f is Banach–Mazur computable, i.e. if f maps computable sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$ onto computable sequences $\{f(t_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$.

(b) f is effective uniformly continuous, i.e. if there is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that for every $N \in \mathbb{N}$ and all $t_1, t_2 \in \mathbb{T}$ with $|t_1 - t_2| \leq 1/d(N)$ always $|f(t_1) - f(t_2)| \leq 2^{-N}$ is satisfied.

Lemma (equivalent definition of computability):
A function $f : \mathbb{T} \to \mathbb{R}$ is computable if and only if there exists a computable sequence of rational polynomials $\{p_m\}_{m\in\mathbb{N}}$ which *converges effectively* to f in the uniform norm, i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $t \in \mathbb{T}$ and every $N \in \mathbb{N}$

$$m \geq e(N) \qquad \text{implies} \qquad \left|f(t) - p_m(t)\right| \leq 2^{-N} .$$

Remark:

- There exist various notions of computability e.g. *Borel- or Markov computability*.
- Banach–Mazur computability is the weakest form of computability.
  $\Rightarrow$ If a function is not Banach–Mazur computable then it is not computable with respect to any other notion of computability.

Computable Functions in Banach Spaces

- We consider 2π-periodic functions on $\mathbb{R}$.
- We write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ for the additive quotient group of real numbers modulo 2π (think of $\mathbb{T} = [-\pi, \pi)$).
- Let $\mathscr{X}$ be a Banach space of functions on $\mathbb{T}$ with norm $\|f\|_{\mathscr{X}}$.

Definition: A function $f \in \mathscr{X}$ is said to be *$\mathscr{X}$–computable* if

(a) f is computable (i.e. effectively approximable by rational polynomials p_m).

(b) its norm $\|f\|_{\mathscr{X}}$ is computable $\Rightarrow \|f - p_m\|_{\mathscr{X}}$ converges to zero effectively as $m \to \infty$.

The set of all $\mathscr{X}$-computable functions is denoted by $\mathscr{X}_{\mathrm{c}}$.

For continuous functions $\mathscr{C}(\mathbb{T})$, computability implies $\mathscr{C}(\mathbb{T})$-computability.

Lemma:

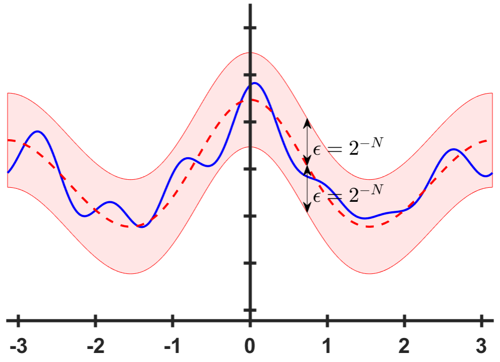
Let $f : \mathbb{T} \to \mathbb{R}$ be a computable function. Then f is computable as a continuous function, i.e. $f \in \mathscr{C}_{\mathrm{c}}(\mathbb{T})$.

J. Avigad and V. Brattka, "Computability and analysis: The legacy of Alan Turing," in *Turing's legacy: developments from Turing's ideas in logic*, ser. Lecture Notes in Logic, Bd. 42. New York: Cambridge University Press, 2014, pp. 1–47.

K. Weihrauch, *Computable Analysis*. Berlin: Springer-Verlag, 2000.

Computability of the Output– Intuition

- The output $i(t) = (\mathrm{T}u)(t)$ of T is usually not explicitly known.
- A function $i(t)$ is computable if it can be *approximated effectively* by a function $p_M(t)$ which can *perfectly be calculated* on a digital computer.
  - $p_M(t)$ might be a rational polynomial of appropriate degree M
  - effective approximation $\Rightarrow$ one can control the approximation error



Computability (an informal definition)

The output $i(t) = (\mathrm{T}u)(t)$ is computable if there exists an algorithm with the following properties

- It can be implemented on a digital computer (a Turing machine).
- It has two inputs: 1. the input $u(t)$ of T 2. an error bound $\varepsilon > 0$.
- It is able to determine in finitely many steps an approximation $p_M(t)$ of $i(t)$ such that the true $i(t)$ is guaranteed to be close to $p_M(t)$, i.e. such that

$$i \in \{f \in \mathscr{X} \ : \ \|f - p_M\|_{\mathscr{X}} < \varepsilon\}$$

where $\mathscr{X}$ is an appropriate Banach space with a corresponding norm $\|\cdot\|_{\mathscr{X}}$.

The Ideal Capacitor

Ideal Capacitor – Negative Result

- We consider the system T given by the voltage-current relation on an ideal capacitor with capacitance C.
- Applying a time-variant voltage $u(t)$ to a capacitor, the corresponding current $i(t)$ is known to be given by

$$i(t) = (\mathrm{T}u)(t) = C\frac{\mathrm{d}u}{\mathrm{d}t}(t) = C\,u'(t)\,, \qquad t \in \mathbb{R}\,. \tag{1}$$

- $C \in \mathbb{R}_{\mathrm{c}}$ is a computable real number $\Rightarrow$ the „system T is commutable“.
- Input signals: $u \in \mathscr{C}^1(\mathbb{T})$, i.e. 2π-periodic continuously differentiable functions on $\mathbb{T} \Rightarrow$ admissible.

Question: Let $u \in \mathscr{C}^1(\mathbb{T}) \cap \mathscr{C}_{\mathrm{c}}(\mathbb{T})$ be an arbitrary admissible input signal for the system T which is additionally computable. Is it true that also the output $i \in \mathscr{C}(\mathbb{T})$ is a computable continuous function?

Theorem: There exists an $u \in \mathscr{C}^1(\mathbb{T})$ with the following properties

1. $u \in \mathscr{C}_c(\mathbb{T})$, i.e. u is a computable continuous function.
2. $u' \in \mathscr{C}(\mathbb{T})$ is absolute continuous and u' has an absolute converging Fourier series.
3. $i(0) = C\frac{\mathrm{d}u}{\mathrm{d}t}(0) = Cu'(0) \notin \mathbb{R}_{\mathrm{c}}$, i.e. the value of the output current at $t = 0$ is not computable.

Ideal Capacitor – Remarks and Further Questions

- Proof: Explicit construction of $u \in \mathscr{C}^1(\mathbb{T})$ such that $u'(0)$ is not computable.
- Similar result for the ideal inductor: $u(t) = Li'(t)$.
- Every non-trivial circuit contains capacitors or inductors.

Previous result holds for a subset $\mathscr{S} \subset \mathscr{C}^1(\mathbb{T}) \cap \mathscr{C}_{\mathrm{c}}(\mathbb{T})$ of all admissible and computable inputs signals (namely for those u for which u' is additionally absolute continuous and possesses an absolute converging Fourier series).

Question: Can we find (large) subsets $\mathscr{B} \subset \mathscr{C}^1(\mathbb{T}) \cap \mathscr{C}_{\mathrm{c}}(\mathbb{T})$ of all admissible and computable input signals such that for every $u \in \mathscr{B}$ the output $i(t) = (\mathrm{T}u)(t)$ of the ideal capacitor is guarenteed to be computable?

We present two sharp characterizations of such subsets:

1. in terms of the second derivative u''
2. in terms of the smoothness of u (in the Sobolev scale)

Good Input Set – In Terms of Second Derivative

Theorem: Let u be the input signal for the ideal capacitor with the following properties

1. $u \in \mathscr{C}^1(\mathbb{T}) \cap \mathscr{C}_c(\mathbb{T})$.
2. u' is absolute continuous and $u'' \in L^1_c(\mathbb{T})$.

Then the output current $i(t) = (\mathrm{T}u)(t) = C\,u'(t)$ is a computable continuous function, i.e. $i \in \mathscr{C}_c(\mathbb{T})$.

Remark:

- If the second derivative of the input signal u belongs to $L^1_c(\mathbb{T})$, then the output i is computable.
- The statement is sharp with respect to the requirement $u'' \in L^1_c(\mathbb{T})$.
  If $u'' \notin L^1_c(\mathbb{T})$ then the output i might not be computable.

Good Input Set – In Terms of Smoothness

- Let $s \in \mathbb{R}_c$, $s \geq 0$ be a computable number.
- For every $f \in L^2(\mathbb{T})$ with its Fourier series $f(t) = a_0/2 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt)$, we define the seminorm

$$\|f\|_{s,2} = \left(\sum_{k=1}^{\infty} k^{2s} \left[|a_k|^2 + |b_k|^2 \right] \right)^{1/2} . \tag{2}$$

- The set $H^s(\mathbb{T}) = \left\{ f \in \mathscr{C}(\mathbb{T}) : \|f\|_{s,2} < \infty \right\}$ equipped with the norm $\|f\|_{H^s(\mathbb{T})} = \max\left(\|f\|_\infty, \|f\|_{s,2} \right)$ becomes a Banach space.
- Parameter s characterizes the smoothness of the functions in $H^s(\mathbb{T})$.

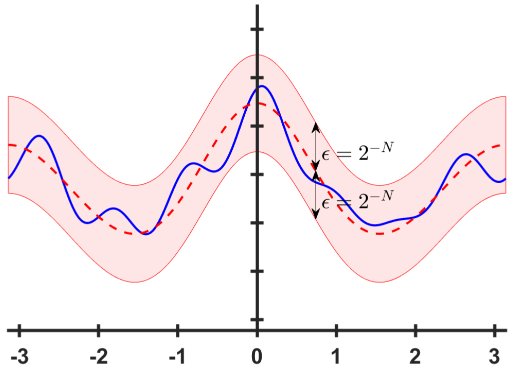
Theorem: Let $\mathrm{T} : u \mapsto i$ be the LTI system given by the ideal capacitor with a computable capacitance $C \in \mathbb{R}_c$, $C > 0$. Then for every $0 \leq s \leq 3/2$ there exists a computable input signal $u \in H^s_c(\mathbb{T})$ such that the output signal i is not computable.

Theorem: Let $s \in \mathbb{R}_c$, $s > 3/2$ and assume $u \in H^s_c(\mathbb{T})$, then $u' \in \mathscr{C}_c(\mathbb{T})$ and so $i = Cu' \in \mathscr{C}_c(\mathbb{T})$.

Summary and Outlook

- ▷ There is generally no closed form expression for the output $u(t) = (\mathrm{T}i)(t)$ of an LTI system.
  $\Rightarrow$ Numerically approximation methods (on digital computers) are applied to determine $u(t)$.

- ▷ Numerically approximation:
  Given input i and $\varepsilon > 0$, determine (in finite time) a confidence interval of width 2ε in which the (unknown) unknown output $u(t)$ lies. $\Rightarrow$ $u(t)$ is computable.



- ▷ Main result: For the ideal capacitor, there exist admissible and computable inputs $i(t)$ such that the corresponding output $u(t) = (\mathrm{T}i)(t)$ is not computable.

- ▷ Good input sets: We characterized sets $\mathscr{B}$ of admissible and computable inputs such that for every $i \in \mathscr{B}$ the corresponding output $u(t) = (\mathrm{T}i)(t)$ is computable.

- ▷ Outlook: Investigation of other systems, characterization of good input sets.