Reconstructing Non-point Sources of Diffusion Fields From Sensor Measurements

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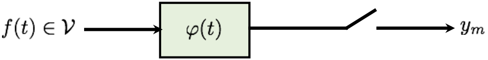
Outline

Inverse Problems

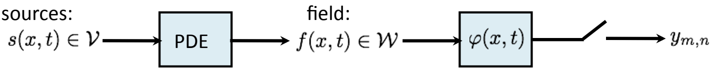
We consider *physics-driven* Inverse Problems

Traditional Sampling Set-up:



-The signal f(t) lies in a subspace, is sparse (e.g., CS), is parametric (e.g., FRI)
-The acquisition device given by the set-up or by design (e.g., random matrix)

Sampling physical fields:



- No assumption on the field but on the sources,
- The acquisition device performs only temporal filtering, no spatial filtering

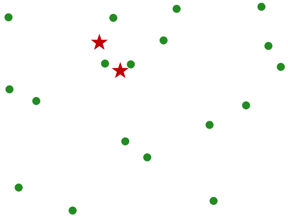
Inverse Problems in Physics: Diffusion

Diffusion

Stochastic movement of a collection of particles from regions of high concentration to regions of lower concentration (until an equilibrium is established).

Sensor networks measure:

- Leakages in/from factories,
- Temperature in server rooms,
- Nuclear fallouts (Fukushima).



The field $u(\mathbf{x}, t)$ induced by a source distribution $f(\mathbf{x}, t)$ satisfies:

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) - \mu \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t). \tag{1}$$

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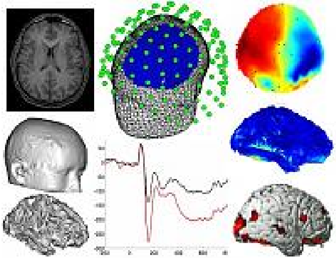
Inverse Problems in Physics: Wave

Wave

A disturbance that travels through a medium from one location to another (transferring energy).

Such fields arise in acoustics, electromagnetics, fluid dynamics and so on. Sensor networks measure:

- Bioelectric neural currents in neurons of cerebral cortex (EEG/MEG),
- Pressure waves from a speaker/acoustic source.



$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\mathbf{x}, t) - \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t).$$

Other PDEs: Laplace's Equation, Advection-/Convection-Diffusion Equation, Helmholtz and many more.

Given these (spatiotemporal) measurements we may wish to find:

- source of factory leakage, detect plume sources
- find hot/cold spots in server clusters
- predict nuclear fallout concentration elsewhere
- center of mass of active regions
- acoustic source localization

Sources can be localized or non-localized $\longrightarrow$ Parameterize sources f.

Problem Formulation: Field Sources

	Instantaneous	Non-Instantaneous
Point	$f(\mathbf{x},t)=\sum_{m=1}^{M} c_m \delta(\mathbf{x}-\boldsymbol{\xi}_m, t-\tau_m)$	$f(\mathbf{x},t)=\sum_{m=1}^{M} c_m e^{\alpha_m(t-\tau_m)}\delta(\mathbf{x}-\boldsymbol{\xi}_m)H(t-\tau_m)$
Line	$f(\mathbf{x},t)=cL(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t)=cL(\mathbf{x})e^{\alpha(t-\tau)}H(t-\tau)$
Polygonal	$f(\mathbf{x},t)=cF(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t)=cF(\mathbf{x})e^{\alpha(t-\tau)}H(t-\tau)$

Where,

- $L(\mathbf{x}) \in \Omega$ describes a line with endpoints $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$.
- $F(\mathbf{x}) \in \Omega$ describes a convex polygon with vertices $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_M\}$.
- $\alpha_m, c_m, \boldsymbol{\xi}_m$ and τ_m is the release rate, intensity, location and activation time of m-th source.

Problem Formulation: Field PDE Model

Let $u(\mathbf{x}, t)$ denote the field induced by a source distribution $f(\mathbf{x}, t)$ then a physics-driven system, in general, has the Green's function solution:

$$u(\mathbf{x}, t) = (f * g)(\mathbf{x}, t) = \int_{\mathbf{x}' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x} - \mathbf{x}', t - t') \, \mathrm{d}t' \mathrm{d}\mathbf{x}' \quad (2)$$

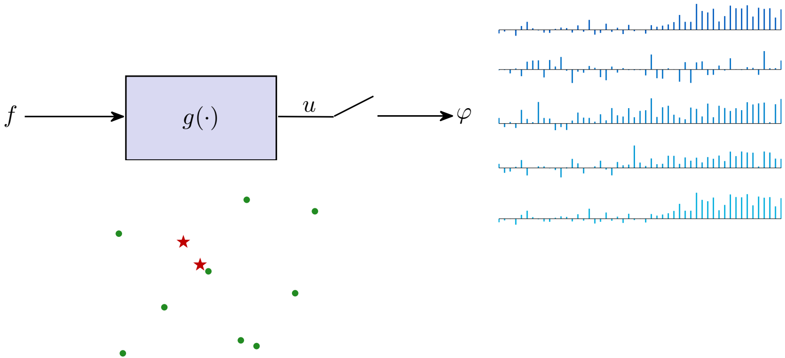
where $g(\mathbf{x}, t)$ is the Green function of the field.
For e.g.,

- **2D diffusion field:** $\frac{\partial}{\partial t} u(\mathbf{x}, t) - \mu \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t)$, has $g(\mathbf{x}, t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$, where $H(t)$ is the step function.

Problem Formulation: Field Measurements

Aim

Estimate $f(\mathbf{x}, t)$ from spatiotemporal samples $\{\varphi_{n,l} = u(\mathbf{x}_n, t_l)\}_{n,l}$ for $n = 1, \ldots, N$ and $l = 0, \ldots, L$, of the measured field.



Source Reconstruction Framework

Recall that

$$u(\mathbf{x}, t) = \int_{\mathbf{x}' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x} - \mathbf{x}', t - t') \, \mathrm{d}t' \mathrm{d}\mathbf{x}'$$
$$= \langle f(\mathbf{x}', t'), g(\mathbf{x} - \mathbf{x}', t - t') \rangle_{\mathbf{x}', t'} .$$

Mathematically the spatiotemporal sample $\varphi_{n,l}$ is

$$\varphi_{n,l} = u(\mathbf{x}_n, t_l)$$
$$= \langle f(\mathbf{x}, t), g(\mathbf{x}_n - \mathbf{x}, t_l - t) \rangle_{\mathbf{x}, t} \quad (3)$$

Consider a weighted-sum of the samples $\{\varphi_{n,l}\}_{n,l}$:

$$\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}\varphi_{n,l} = \sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}\left\langle f(\mathbf{x},t), g(\mathbf{x}_n-\mathbf{x}, t_l-t)\right\rangle_{\mathbf{x},t}$$
$$= \left\langle f(\mathbf{x},t), \underbrace{\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l} g(\mathbf{x}_n-\mathbf{x}, t_l-t)}_{=\Psi_k(\mathbf{x})\Gamma(t)}\right\rangle, \qquad (4)$$

where $w_{n,l} \in \mathbb{C}$ are some arbitrary weights (to be determined).
We wish to find $f(\mathbf{x},t)$:

- For our source types, can we choose functions $\Psi_k(\mathbf{x})$ and $\Gamma(t)$ that makes this problem tractable? — YES!

Let these (new) *generalized measurements* be

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \varphi_{n,l} = \langle f(\mathbf{x}, t), \Psi_k(\mathbf{x}) \Gamma(t) \rangle$$
$$= \int_{\Omega} \int_{t \in [0,T]} \Psi_k(\mathbf{x}) \Gamma(t) f(\mathbf{x}, t) \mathrm{d}t \mathrm{d}V,$$

where $\Psi_k(\mathbf{x})$ for $k \in \mathbb{Z}^d$, $d = \{1, 2\}$, and $\Gamma(t)$ a family of properly chosen *spatial* and *temporal sensing functions*, respectively.
Proper choice $\Longrightarrow$ solvability & stability of new problem.

- As an example, take the **instantaneous** source distribution $f(\mathbf{x}, t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m)$, then:

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m \Psi_k(\boldsymbol{\xi}_m) \Gamma(\tau_m).$$

Choice of Sensing Functions: 2D Case

For $\mathbf{x} \in \mathbb{R}^2$, we may choose

- $\Gamma(t) = e^{-\mathrm{j}t/T}$, and
- $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$, for $k = 0, 1, \ldots, K$.

Then,

$$\begin{aligned} \mathcal{R}(k) &= \sum_{m=1}^{M} c_m e^{-\mathrm{j}\tau_m/T} e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})} \\ &= \sum_{m=1}^{M} c'_m v_m^k. \end{aligned}$$

Can be solved to jointly recover $c'_m = c_m e^{-\mathrm{j}\tau_m/T}$ and $v_m = e^{-(\xi_{1,m}+\mathrm{j}\xi_{2,m})}$ using Prony's method for $m = 1, \ldots, M$ providing $K \geq 2M - 1$.

Line Source

Instantaneous case: $f(\mathbf{x}, t) = cL(\mathbf{x})\delta(t-\tau)$, thus $\mathcal{R}(k)$ reduces to:

$$\begin{aligned}\mathcal{R}(k) &= \int_{\Omega}\int_{t} \Psi_k(\mathbf{x})\Gamma(t) f(\mathbf{x}, t)\,\mathrm{d}t\,\mathrm{d}V \\ &= c\Gamma(\tau)\int_{\Omega} \Psi_k(\mathbf{x}) L(\mathbf{x})\,\mathrm{d}V \\ &= c\Gamma(\tau)\int_{L(\mathbf{x})} \Psi_k(\mathbf{x})\,\mathrm{d}S \\ &= \frac{1}{k} c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\Gamma(\tau)\sum_{m=1}^{2}(-1)^m \Psi_k(\boldsymbol{\xi}_m)\end{aligned}$$

Line source

From $\mathcal{R}(k) = \frac{1}{k} c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\boldsymbol{\xi}_m)$ and the usual choice for sensing functions $\Gamma(t) = e^{-\mathrm{j}t/T}$ and $\Psi_k(\mathbf{x}) = e^{-k(x_1 + \mathrm{j}x_2)}$, then:

$$\mathcal{R}'(k) \triangleq k\mathcal{R}(k) = c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\boldsymbol{\xi}_m)$$

$$= c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) e^{-\mathrm{j}\tau/T} \sum_{m=1}^{2} (-1)^m e^{-k(\xi_{1,m} + \mathrm{j}\xi_{2,m})}$$

Can again recover c, τ and the endpoints $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ of the line source using Prony's method (providing $K \geq 3$).

- For polygonal sources:
  surface integral → line integral → Ψ_k **evaluated at vertices**

Computing $\mathcal{R}(k)$ reliably from sensor Measurements?

Recall that,

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \varphi_{n,l}$$

Thus computing $\mathcal{R}(k)$ is equivalent to finding the weights $w_{n,l}$. These weights may be found:

1. Using Green's second identity
   - For 2D diffusion field.
2. Formulating and solving a linear system (**explicitly**)
   - Inversion of large matrices.
   - Conditioning and stability considerations.
3. Results from non-uniform/universal sampling theory?
   - Stable iterative/non-iterative algorithms.

Explicit computation of weights $\{w_{n,l}\}_{n,l}$

We desire $\{w_{n,l}\}_{n,l}$, so that $\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l} g(\mathbf{x}_n - \mathbf{x}, t_l - t) = \Psi_k(\mathbf{x})\Gamma(t)$, where g, Ψ_k and Γ are known.

For e.g. the 2D heat problem $g(\mathbf{x}, t) = \frac{1}{4\pi t} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$, and we may choose, $\Psi_k(\mathbf{x}) = e^{-k(x_1 + \mathrm{j}x_2)}$ and $\Gamma(t) = e^{-\mathrm{j}t/T}$.

Can formulate a linear system as follows:

$$\begin{bmatrix} g(\mathbf{x}_1 - \mathbf{x}'_1, t_l - t_j) & \cdots & g(\mathbf{x}_N - \mathbf{x}'_1, t_l - t_j) \\ \vdots & & \vdots \\ g(\mathbf{x}_1 - \mathbf{x}'_I, t_l - t_j) & \cdots & g(\mathbf{x}_N - \mathbf{x}'_I, t_l - t_j) \end{bmatrix} \begin{bmatrix} w_{1,l} \\ \vdots \\ w_{N,l} \end{bmatrix} = \begin{bmatrix} \Psi_k(\mathbf{x}'_1)\Gamma(t_j) \\ \vdots \\ \Psi_k(\mathbf{x}'_I)\Gamma(t_j) \end{bmatrix}$$

$$\mathbf{G}_{l,j}\mathbf{w}_l = \mathbf{p}_j$$

$$\Rightarrow \begin{bmatrix} \mathbf{G}_{0,1} & \cdots & \mathbf{G}_{0,J} \\ \vdots & & \vdots \\ \mathbf{G}_{L,1} & \cdots & \mathbf{G}_{L,J} \end{bmatrix}^\top \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_J \end{bmatrix} = \begin{bmatrix} \Psi_k(\mathbf{x}'_1)\Gamma(t_1) \\ \vdots \\ \Psi_k(\mathbf{x}'_I)\Gamma(t_J) \end{bmatrix}.$$

$$\mathbf{G}\mathbf{w} = \mathbf{p}$$

Solve $\mathbf{G}\mathbf{w} = \mathbf{p}$, where $\mathbf{G} \in \mathbb{R}^{N(L+1)\times IJ}$, $\mathbf{w} \in \mathbb{R}^{N(L+1)}$ and $\mathbf{p} \in \mathbb{R}^{IJ}$.

Implicit computation of weights $\{w_{n,l}\}_{n,l}$

1. **Green's second identity:** Let $u(\mathbf{x}, t)$ and $\Psi_k(\mathbf{x})$ be scalar functions in $\mathcal{C}^2$, over $\Omega \in \mathbb{R}^2$, then:

$$\oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S = \int_{\Omega} \left(\Psi_k \nabla^2 u - u \nabla^2 \Psi_k\right) \mathrm{d}V,$$

   where $\hat{\mathbf{n}}_{\partial\Omega}$ is the outward pointing unit normal to the boundary $\partial\Omega$.

2. Substitute (inhomogenous) PDE and choose Ψ_k to satisfy $\frac{\partial \Psi_k}{\partial t} + \mu \nabla^2 \Psi_k = 0$, thus:

$$\int_{\Omega} \frac{\partial}{\partial t}(u\Psi_k)\mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S = \int_{\Omega} \Psi_k f \mathrm{d}V.$$

3. Multiply through by $\Gamma(t)$ and integrate over $t = [0, T]$:

$$\underbrace{\int_0^T \Gamma \int_{\Omega} \Psi_k \frac{\partial u}{\partial t} + u \frac{\partial \Psi_k}{\partial t} \mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S \mathrm{d}t}_{=\mathcal{R}(k)} = \int_{\Omega} \int_0^T \Psi_k \Gamma f \,\mathrm{d}t \,\mathrm{d}V$$

Implicit computation of weights $\{w_{n,l}\}_{n,l}$

From:

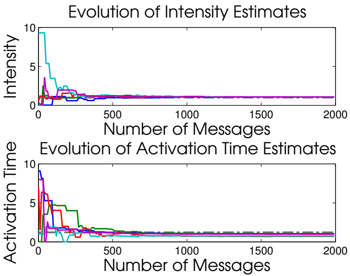
$$\underbrace{\int_0^T \Gamma \int_\Omega \Psi_k \frac{\partial u}{\partial t} + u \frac{\partial \Psi_k}{\partial t} \mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S \mathrm{d}t}_{=\mathcal{R}(k)} = \int_\Omega \int_0^T \Psi_k \Gamma f \,\mathrm{d}t\,\mathrm{d}V$$

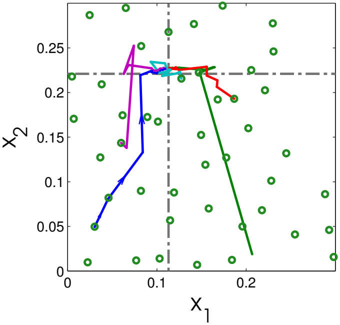
$$\Rightarrow \mathcal{R}(k) = \langle f(\mathbf{x}, t), \Psi_k(\mathbf{x})\Gamma(t) \rangle$$

As such we can obtain $\{\mathcal{R}(k)\}$ by **approximating the integrals** from the spatiotemporal samples using standard quadrature schemes.

- Mesh required.
- Integral simply a linear combination of field samples.
- Distributed computation (consensus-based estimation).

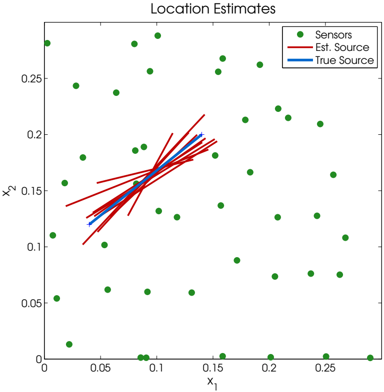
Synthetic data: Point Diffusion Source

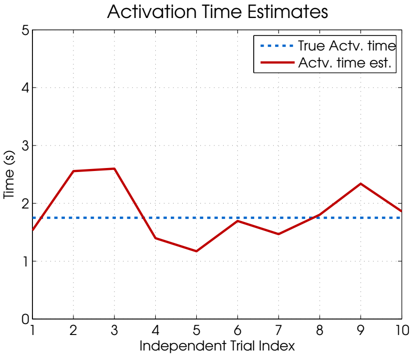




Distributed estimation for $M = 1$ source using 45 sensors, field is sampled for $T_{end} = 10s$ at $\frac{1}{\Delta t} = 1Hz$. $K = 1$.

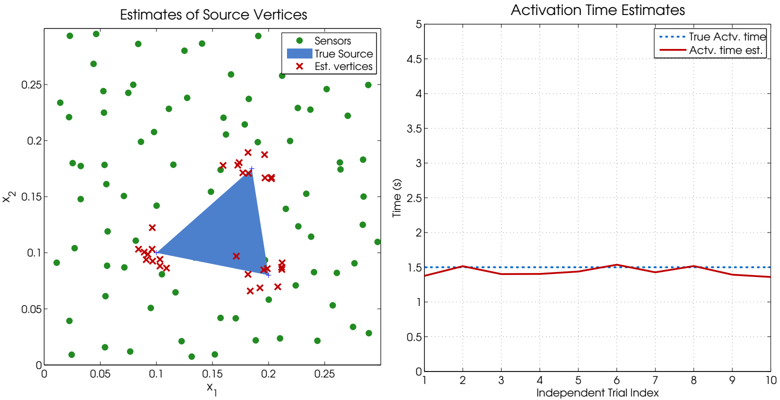
Synthetic data: Line Diffusion Source





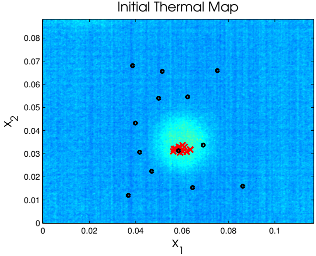
$N = 45$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement SNR$= 20$dB. $K = 6$ and $R = 5$.

Synthetic data: Triangular Diffusion Source

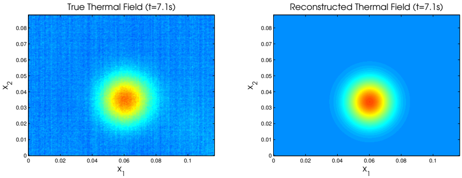


$N = 90$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement SNR$= 35$dB. $K = 6$ and $R = 5$.

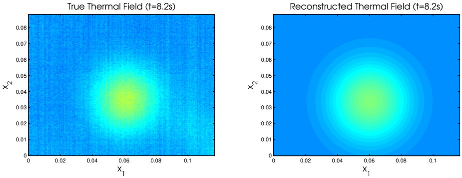
Simulation Results: Real Diffusion Data



(a) Thermal distribution (immediately after activation) and location estimates.



(b) Real field (left) and its reconstruction (right) at $t = 7.1s$.



(c) Real field (left) and its reconstruction (right) at $t = 8.2s$.

Simulation Results: Laplace - Synthetic data

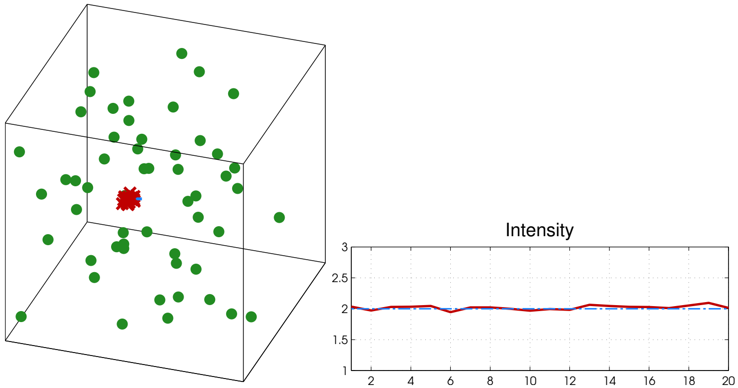


Figure 1: Single point source recovery in 3D using samples obtained by $N = 57$ sensors with $K_1 = K_2 = 1$ for spatial sensing function family. Results for 20 independent trials are given.

Further Extensions

1. Reconstructing non-localized sources: line and (convex) polygons.
   - Compute generalized measurements.
   - Use tools from complex analysis to modify $\mathcal{R}(k)$.
   - Recover endpoints (vertices) of line (polygonal) source.

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3. **Generalisation Possible?**
   - Same principle can be generalized to PDE-driven fields: wave, Poisson etc.
   - How to compute the field analysis coefficients $\{w_{n,l}\}$?
   - Turn to FRI theory: exponential reproduction.

Thank You.