On the frequency domain detection of high dimensional time series

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Introduction - Setting

M–dimensional complex time series $(\mathbf{y}_n)_{n\in\mathbb{Z}}$ modeled as

$$\mathbf{y}_n = \underbrace{\sum_{k=0}^{+\infty} \mathbf{H}_k \boldsymbol{\epsilon}_{n-k}}_{=\mathbf{u}_n} + \mathbf{v}_n \in \mathbb{C}^M$$

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$\mathcal{H}_0 : \mathbf{S}_\mathbf{y}(\nu) = \mathrm{diag}(\mathbf{S}_\mathbf{y}(\nu)) = \mathbf{S}_\mathbf{v}(\nu)$ (noise only) vs
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With $\mathbf{C}_{\mathbf{y}}(\nu) := \mathrm{diag}(\mathbf{S}_{\mathbf{y}}(\nu))^{-\frac{1}{2}} \mathbf{S}_{\mathbf{y}}(\nu) \mathrm{diag}(\mathbf{S}_{\mathbf{y}}(\nu))^{-\frac{1}{2}}$

Frequency domain detection hypothesis test - $\mathbf{C}_{\mathbf{y}}$

$\mathcal{H}_0 : \mathbf{C}_{\mathbf{y}} = \mathbf{I}_M$ (pure noise) vs $\mathcal{H}_1 : \mathbf{C}_{\mathbf{y}} \neq \mathbf{I}_M$ (signal + noise). Use frequency domain estimators of $\mathbf{C}_{\mathbf{y}}$ to test if $\mathbf{u}_n = 0$.

Introduction - Signal detection context

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- finite $K \times \mathcal{O}(1)$ signal eigenvalues vs $M \times \mathcal{O}(1)$ noise eigenvalues
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- Considerable work still needed for **dynamic / wideband models**
- Temporal approaches also possible, but frequency ones turns out to be simpler.

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Smoothed periodogram estimator of the spectral density matrix :

$$\hat{\mathbf{S}}_\mathbf{y}(\nu) = \frac{1}{B+1} \sum_{b=-B/2}^{B/2} \xi_\mathbf{y}\left(\nu + \frac{b}{N}\right) \xi_\mathbf{y}\left(\nu + \frac{b}{N}\right)^* \quad (B : \text{smoothing span})$$

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Estimator of the spectral coherency matrix :

$$\hat{\mathbf{C}}_{\mathbf{y}}(\nu) = \mathrm{diag}(\hat{\mathbf{S}}_{\mathbf{y}}(\nu))^{-\frac{1}{2}} \hat{\mathbf{S}}_{\mathbf{y}}(\nu) \mathrm{diag}(\hat{\mathbf{S}}_{\mathbf{y}}(\nu))^{-\frac{1}{2}}$$

Main result on $\hat{\mathbf{C}}_\mathbf{y}$

High dimensional regime : consider $B := B(N)$, $M := M(N)$ such that

$$M, B, N \xrightarrow{N\to\infty} +\infty, \quad \frac{B}{N} \xrightarrow{N\to\infty} 0, \quad \frac{M}{B} \xrightarrow{N\to\infty} c \in (0,1)$$

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Theorem - Wishart approximation of $\hat{\mathbf{C}}_\mathbf{y}$

Under proper technical assumptions on the signal and noise, there exists a $M \times (B+1)$ random matrix $\mathbf{X}(\nu)$ with i.i.d. $\mathcal{N}_\mathbb{C}(0,1)$ entries such that

$$\max_{\nu \in \mathcal{V}_N} \left\| \hat{\mathbf{C}}_\mathbf{y}(\nu) - \mathbf{\Xi}(\nu)^{\frac{1}{2}} \frac{\mathbf{X}(\nu)\mathbf{X}(\nu)^*}{B+1} \mathbf{\Xi}(\nu)^{\frac{1}{2}} \right\| \xrightarrow[N\to\infty]{a.s.} 0 \tag{1}$$

where $\mathbf{\Xi}(\nu) = \underbrace{\mathbf{S}_\mathbf{v}(\nu)^{-\frac{1}{2}}\mathbf{H}(\nu)\mathbf{H}(\nu)^*\mathbf{S}_\mathbf{v}(\nu)^{-\frac{1}{2}}}_{\text{rank } K<M} + \mathbf{I}_M$ and $\mathbf{H}(\nu) := \sum_{k=0}^{+\infty} \mathbf{H}_k \mathrm{e}^{-\mathrm{i}2\pi\nu k}$

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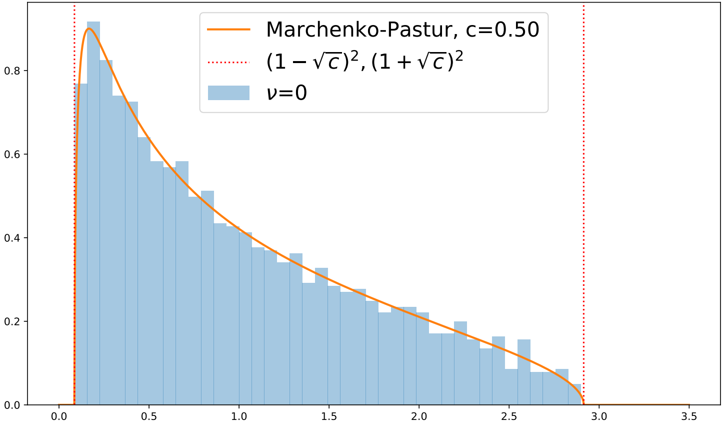
Key idea :

- $\mathbf{\Xi}(\nu)$ fixed rank K perturbation of the identity matrix. This is not the case with temporal approaches.
- first order behaviour of $\mathbf{\Xi}(\nu)^{\frac{1}{2}} \frac{\mathbf{X}(\nu)\mathbf{X}(\nu)^*}{B+1} \mathbf{\Xi}(\nu)^{\frac{1}{2}}$ known.

Simulation in the pure noise case ($K = 0$)

- $K = 0$ ($\mathbf{y}_n = \mathbf{v}_n$ as MA(1)), $M = 100$, $B = 200$, $N = 4000$
- asymptotically, eigenvalues of $\hat{\mathbf{C}}_{\mathbf{y}}(\nu) \in [(1-\sqrt{c})^2, (1+\sqrt{c})^2]$ (Marchenko & Pastur, 1967)
- good fit even for small dimensions (20 realisations)



Application - Spectral behaviour of $\hat{\mathbf{C}}_{\mathbf{y}}$

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Assumption - Spike

For all $k \in \{1,\ldots,K\}$, there exists $\gamma_k > 0$ such that

$$\lambda_k\left(\mathbf{S}_\mathbf{v}(\nu_N^*)^{-\frac{1}{2}}\mathbf{H}(\nu_N^*)\mathbf{H}(\nu_N^*)^*\mathbf{S}_\mathbf{v}(\nu_N^*)^{-\frac{1}{2}}\right) \xrightarrow[N\to\infty]{} \gamma_k$$

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Corollary - Behaviour of the spectrum of $\hat{\mathbf{C}}_{\mathbf{y}}(\nu)$

Under proper technical assumptions, for all $k = 1,\ldots,K$ and all $\nu \in \mathcal{V}_N$,

$$\lambda_k\left(\hat{\mathbf{C}}_{\mathbf{y}}(\nu_N^*)\right) \xrightarrow[N\to\infty]{a.s.} \begin{cases} \frac{(\gamma_k+1)(\gamma_k+c)}{\gamma_k} > (1+\sqrt{c})^2 & \text{if } \gamma_k > \sqrt{c} \\ (1+\sqrt{c})^2 & \text{if } \gamma_k \le \sqrt{c} \end{cases}$$

whereas

$$\lambda_{K+1}\left(\hat{\mathbf{C}}_{\mathbf{y}}(\nu_N^*)\right) \xrightarrow[N\to\infty]{a.s.} (1+\sqrt{c})^2$$

Application - Spectral behaviour of $\hat{\mathbf{C}}_\mathbf{y}$ - Simulation

- rank one signal ($\mathbf{h}(\nu)$ vector) + M–dimensional noise MA(1) process.
- K=1, same $M = 100$, $B = 200$, $N = 4000$, $c = 0.5 \implies \sqrt{c} \approx 0.7$
- separation starting at $SNR := \gamma_1 = \sqrt{c} \implies$ detection for low frequencies

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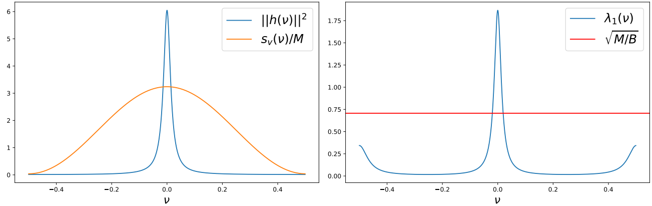

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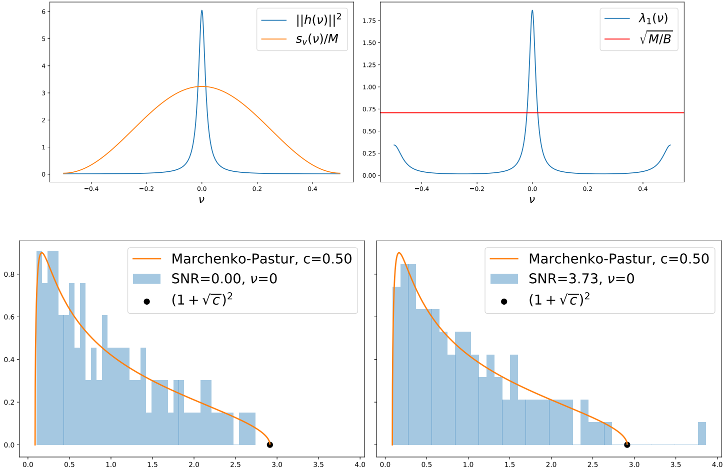

Application - Simulation - Varying parameters

- $M = 20, B = 40, N = 4000, c = 0.5,$ ma parameter $= 0.6,$ medium SNR.
- as $\frac{B}{N} \to 0$, the finite sample results are closer to the asymptotics

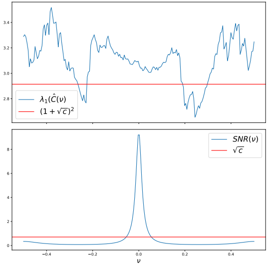

FIGURE – $B/N = 0.5$

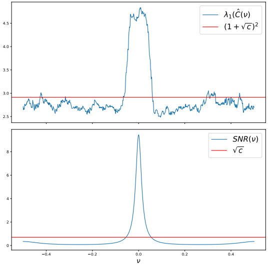

FIGURE – $B/N = 0.1$

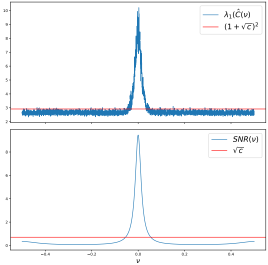

FIGURE – $B/N = 0.01$

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New frequency domain detection algorithm

Consider, for some threshold $\epsilon > 0$ the following procedure :

$$\begin{cases} \lambda_1(\hat{\mathbf{C}}_{\mathbf{y}}(\nu_N^*)) < (1+\sqrt{c})^2 + \epsilon & \text{absence of u is decided} \\ \lambda_1(\hat{\mathbf{C}}_{\mathbf{y}}(\nu_N^*)) > (1+\sqrt{c})^2 + \epsilon & \text{presence of u is decided} \end{cases}$$

This leads to define the test statistics :

$$T_\epsilon = \mathbb{1}_{((1+\sqrt{c})^2+\epsilon,+\infty)}\left(\max_{\nu \in \mathcal{V}_N} \left\| \hat{\mathbf{C}}_{\mathbf{y}}(\nu) \right\| \right)$$

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Theorem - Spectral detection testing

Under proper assumptions, the previous test is consistent iif $\gamma_1 > \sqrt{c}$ and ϵ small enough.

Conclusion

Contributions :

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- Our detection algorithm is based on a phase transition phenomenon of the largest eigenvalues of $\hat{\mathbf{C}}_{\mathbf{y}}(\nu)$:
  - weak energy signals $\Longrightarrow$ eigenvalue absorbed in the noise bulk
  - high energy signals $\Longrightarrow$ eigenvalue separated from the noise bulk