Computing Hilbert Transform and Spectral Factorization for Signal Spaces of Smooth Functions

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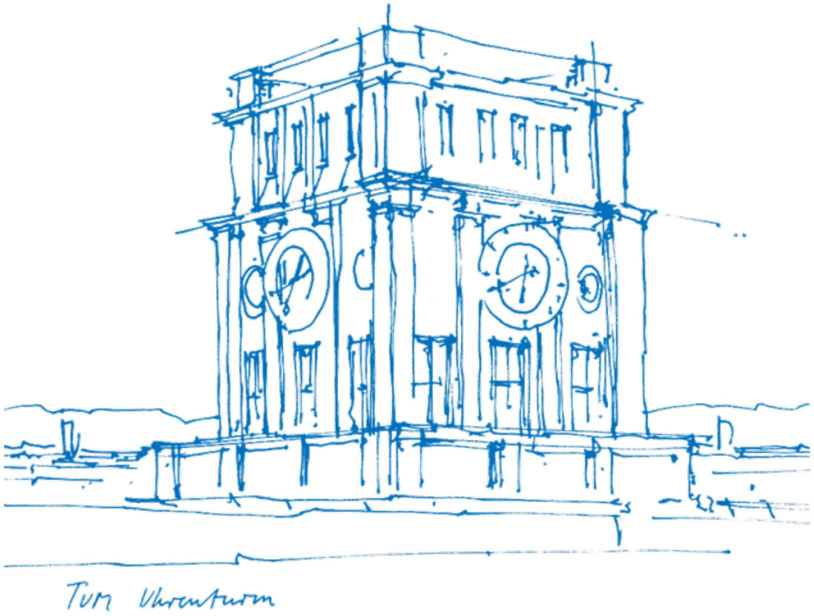


Subject and Outline of the Talk

Is it always possible to calculate the Hilbert transform and the spectral factorization on a digital computer?

Outline

1. Hilbert Transform and Spectral Factorization – A very short Introduction
2. Review of Computability Theory
3. Non-Computability/Computability of the Hilbert Transform and Spectral Factorization
4. Summary and Outlook

Hilbert Transform and Spectral Factorization

The Hilbert Transformation

▷ Let $f \in L^2(\partial\mathbb{D})$ be function on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ with Fourier series

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{with} \qquad c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau$$

▷ With f one associates its conjugate function, defined by

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{H}f)(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\,c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{with} \qquad \mathrm{sgn}(n) = \begin{cases} 0 & : n = 0 \\ n/|n| & : n \neq 0 \end{cases}.$$

▷ The linear mapping $\mathrm{H} : f \mapsto \widetilde{f}$ is known as Hilbert transform and can be written as a principal value integral as

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{H}f)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\mathrm{e}^{\mathrm{i}\tau})}{\tan([\theta-\tau]/2)}\,\mathrm{d}\tau\,, \qquad \theta \in [-\pi,\pi)\,.$$

Application

- Physics: Kramers–Kronig relations
- Real- and imaginary part of a causal signal are related by the Hilbert transform

Spectral Factorization

- ▷ Let ϕ be a spectral density. That is
  - a non-negative real function on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$
  - satisfying the *Paley–Wiener (Szegö) condition* $\log\phi \in L^1(\partial\mathbb{D})$
- ▷ Spectral factorization is the operation of writing ϕ as

$$\phi(\mathrm{e}^{\mathrm{i}\omega}) = \phi_+(\mathrm{e}^{\mathrm{i}\omega})\,\phi_-(\mathrm{e}^{\mathrm{i}\omega}) = \left|\phi_+(\mathrm{e}^{\mathrm{i}\omega})\right|^2 , \qquad \omega \in [-\pi, \pi) .$$

with the spectral factor ϕ_+ and its *para-Hermitian conjugate* $\phi_-(z) = \overline{\phi_+(1/\overline{z})}$ for $z \in \mathbb{C}$.
The spectral factor ϕ_+ is an *outer function* (a "minimum-phase system"), i.e.
  - $\phi_+(z)$ is analytic for every $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
  - $\phi(z) \neq 0$ for all $z \in \mathbb{D}$.
- ▷ The spectral factor can be written as

$$\phi_+(z) = (\mathrm{S}\phi)(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\phi(\mathrm{e}^{\mathrm{i}\omega})\,\frac{\mathrm{e}^{\mathrm{i}\omega} + z}{\mathrm{e}^{\mathrm{i}\omega} - z}\,\mathrm{d}\omega\right) , \qquad z \in \mathbb{D} .$$

- ▷ We call the mapping $\mathrm{S} : \phi \mapsto \phi_+$ the spectral factorization mapping.

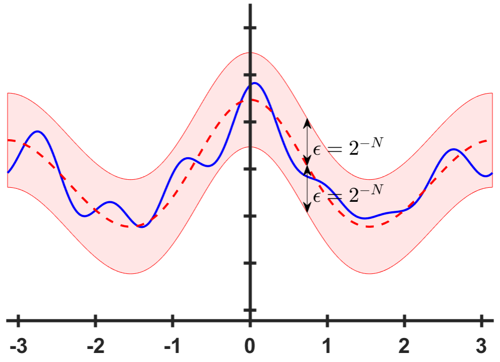
Applications

- Wiener–Kolmogorov theory of smoothing and prediction of stationary time series
- causal Wiener filter: Communications, signal processing, control theory, $\cdots$

Computability

Computability – Intuition

- The true spectral factor ϕ_+ is usually not known explicitly.
- A function ϕ_+ is computable if it can be *approximated effectively* by a function p_M which can *perfectly be calculated* on a digital computer.
  - p_M might be a rational polynomial of a certain degree M
  - effective approximation $\Rightarrow$ control of approximation error



Computability (an informal definition)

The spectral factor ϕ_+ is computable if there exists an algorithm with the following properties

- It can be implemented on a digital computer (a Turing machine).
- It has two inputs: 1. the spectral density ϕ 2. an error bound $\varepsilon > 0$.
- It is able to determine in finitely many steps an approximation p_M of ϕ_+ such that the true ϕ_+ is guaranteed to be close to p_M, i.e. such that

$$\phi_+ \in \{\psi \in \mathscr{X} \ : \ \|\psi - p_M\|_{\mathscr{X}} < \varepsilon\}$$

where $\mathscr{X}$ is an appropriate Banach space with a corresponding norm $\|\cdot\|_{\mathscr{X}}$.

Computable Rational Numbers

Definition: A sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers is said to be computable if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ and such that

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}\,, \qquad n \in \mathbb{N}\,.$$

A recursive function $a : \mathbb{N} \to \mathbb{N}$ is a mapping that is build form elementary computable functions and recursion and can be calculated on a *Turing machine*.

Turing machine

- can simulate any given algorithm and therewith provide a simple but very powerful model of computation.
- is a theoretical model describing the fundamental limits of any realizable digital computer.
- Most powerful programming languages are called Turing-complete (such as C, C++, Java, etc.).

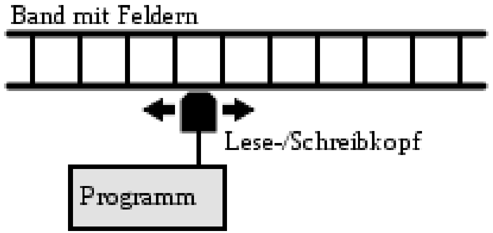


Figure taken from *Wikipedia*

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. s2-42, no. 1, 1937.

Computable Real Numbers

- ▷ Any real number $x \in \mathbb{R}$ is the limit of a sequence of rational numbers.
- ▷ For $x \in \mathbb{R}$ to be computable, the convergence has to be effective.

Definition (Computable number): A real number $x \in \mathbb{R}$ is said to be *computable* if there exists a computable sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers which *converges effectively* to x,
i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$

$$|x - r_n| \leq 2^{-N} \qquad \text{whenever} \quad n \geq e(N)\,.$$

$\Rightarrow$ $x \in \mathbb{R}$ is computable if a Turing machine can approximate it with exponentially vanishing error.

- $\mathbb{R}_c$ stand for the set of all *computable real numbers*.
- $\mathbb{C}_c = \{x + iy : x, y \in \mathbb{R}_c\}$ stands for the set of all *computable complex numbers*.
- Note that the set of computable numbers $\mathbb{R}_c \subsetneq \mathbb{R}$ is only countable.

Computable Functions

Definition: A function $f : \partial\mathbb{D} \to \mathbb{R}$ on the unit circle is said to be computable if

(a) f is Banach–Mazur computable, i.e. if f maps computable sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$ onto computable sequences $\{f(x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$.

(b) f is effective uniformly continuous, i.e. if there is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that for every $N \in \mathbb{N}$ and all $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ with $|\zeta_1 - \zeta_2| \leq 1/d(N)$ always $|f(\zeta_1) - f(\zeta_2)| \leq 2^{-N}$ is satisfied.

Lemma (equivalent definition of computability):

A function $f : \partial\mathbb{D} \to \mathbb{R}$ on the unit circle is computable if and only if there exists a sequence of rational polynomials $\{p_m\}_{m\in\mathbb{N}}$ which *converges effectively* to f in the uniform norm, i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $\theta \in [-\pi, \pi)$ and every $N \in \mathbb{N}$

$$m \geq e(N) \qquad \text{implies} \qquad \left| f(\mathrm{e}^{\mathrm{i}\theta}) - p_m(\mathrm{e}^{\mathrm{i}\theta}) \right| \leq 2^{-N} .$$

Remark:

- There exist various notions of computability e.g. *Borel- or Markov computability*.
- Banach–Mazur computability is the weakest form of computability.
  $\Rightarrow$ If a function is not Banach–Mazur computable then it is not computable with respect to any other notion of computability.

Computable Functions in Banach Spaces

We consider functions in a Banach space $\mathscr{X}$ of functions on $\partial\mathbb{D}$ with norm $\|f\|_{\mathscr{X}}$.

Definition: A function $f \in \mathscr{X}$ is said to be $\mathscr{X}$–*computable* if

(a) f is computable (i.e. effectively approximable by rational polynomials p_m).

(b) its norm $\|f\|_{\mathscr{X}}$ is computable $\Rightarrow$ $\|f - p_m\|_{\mathscr{X}}$ converges to zero effectively as $m \to \infty$.

The set of all $\mathscr{X}$-computable functions is denoted by $\mathscr{X}_{\mathrm{c}}$.

J. Avigad and V. Brattka, "Computability and analysis: The legacy of Alan Turing," in *Turing's legacy: developments from Turing's ideas in logic*, ser. Lecture Notes in Logic, Bd. 42. New York: Cambridge University Press, 2014, pp. 1–47.

K. Weihrauch, *Computable Analysis*. Berlin: Springer-Verlag, 2000.

Non-Computability and Computability of the Hilbert Transformation

The Hilbert Transform is Generally not Computable

There exists computable continuous functions such that its Hilbert transform is not a computable function.

- ▷ Dirichlet energy: For every $f \in \mathscr{C}(\partial\mathbb{D})$ with $f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} c_n(f)\,\mathrm{e}^{in\theta}$ define

$$\|f\|_{\mathrm{E}} = \left(\sum_{n\in\mathbb{Z}} |n|\,|c_n(f)|^2\right)^{1/2}.$$

- ▷ Signal space: Continuous functions of finite Dirichlet energy

$$\mathscr{B} = \left\{f \in \mathscr{C}(\partial\mathbb{D}) \,:\, \|f\|_{\mathrm{E}} < \infty\right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}} = \max\left(\|f\|_\infty, \|f\|_{\mathrm{E}}\right).$$

Theorem: There exist computable functions $f \in \mathscr{B}_{\mathrm{c}}$ so that

1. $\widetilde{f} \in \mathscr{B}$ with $0 < \widetilde{f}(0) < 1$ is absolut continuous
2. f is absolute continuous
3. $f \in \mathscr{W}$ with $\|f\|_{\mathscr{W}} < 1$

but such that $\widetilde{f}(0) = (\mathrm{H}f)(0) \notin \mathbb{R}_{\mathrm{c}}$.

H. Boche and V. Pohl, "On the Algorithmic Solvability of the Spectral Factorization and the Calculation of the Wiener Filter on Turing Machines," *IEEE Intern. Symposium on Inform. Theory (ISIT)*, Paris, France, July 2019, 2459–2463.

Sets of Computable Hilbert Transforms

Questions: What are sufficient conditions of $f \in \mathscr{C}_c(\partial\mathbb{D})$ so that $\widetilde{f}$ is computable?

Answer: $f' \in L^p_c(\partial\mathbb{D})$ with $p > 1$.

Theorem:
Let $f \in \mathscr{C}_c(\mathbb{T})$ be absolute continuous so that there exists an $p \in \mathbb{R}_c$, $p > 1$ such that $f' \in L^p_c(\partial\mathbb{D})$.
Then $\widetilde{f} = \mathrm{H}f$ is a computable continuous function, i.e. $\widetilde{f} \in \mathscr{C}_c(\partial\mathbb{D})$.

Theorem:
There exists an absolute continuous $f \in \mathscr{C}_c(\partial\mathbb{D})$ with $f' \in L^1_c(\partial\mathbb{D})$ so that $f \in \mathscr{W}$ and $\widetilde{f} \in \mathscr{W}$
but such that $\widetilde{f}(0) = (\mathrm{H}f)(0) \notin \mathbb{R}_c$.

Non-Computability and Computability of the Spectral Factorization

Spectral Densities

We are going to show that the spectral factor

$$\phi_+(z) = (\mathrm{S}\phi)(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\phi(\mathrm{e}^{\mathrm{i}\theta})\,\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\tau\right)\,, \qquad z \in \mathbb{D}$$

is *not computable*, even for computable spectral densities ϕ with very nice properties.

Definition (Set $\mathscr{D}$ of nice spectral densities):
A spectral density $\phi \in \mathscr{C}(\partial\mathbb{D})$ is said to belong to the set $\mathscr{D}$, if it has the following properties:

- ϕ is absolute continuous.
- ϕ is strictly positive on $\partial\mathbb{D}$, i.e. $\min_{\zeta\in\partial\mathbb{D}} \phi(\zeta) = s > 0$.
- ϕ belongs to the Wiener algebra $\mathscr{W}$, i.e. ϕ possess an absolutely converging Fourier series
- ϕ has finite Dirichlet energy, i.e. $\|\phi\|_{\mathrm{E}} < \infty$
- The spectral factor ϕ_+ has the same properties as ϕ, i.e.
  ϕ_+ is absolute continuous, in the Wiener algebra $\mathscr{W}$, and has finite Dirichlet energy.

The Non-Computability of the Spectral Factorization

Theorem:
To every computable point $\zeta \in \partial\mathbb{D}$ on the unit circle, there exists a *computable* spectral density $\phi \in \mathscr{D}$ such that $\phi_+(\zeta)$ is not a computable number, i.e. such that $\phi_+(\zeta) \notin \mathbb{C}_c$.

Remark:

- $\phi_+(\zeta)$ is not a computable number $\Rightarrow$ ϕ_+ is not Banach-Mazur computable.
- So ϕ_+ is not computable in any stronger notion of computability.
- Note that the input, i.e the spectral density ϕ is computable. However, the corresponding spectral factor ϕ_+ might not be computable.

H. Boche and V. Pohl, "On the Algorithmic Solvability of the Spectral Factorization and the Calculation of the Wiener Filter on Turing Machines," *IEEE Intern. Symposium on Inform. Theory (ISIT)*, Paris, France, July 2019, 2459–2463.

Computability of Spectral Factorization

Questions: What are sufficient conditions of $\phi \in \mathscr{C}_\mathrm{c}(\partial\mathbb{D})$ so that ϕ_+ is computable?

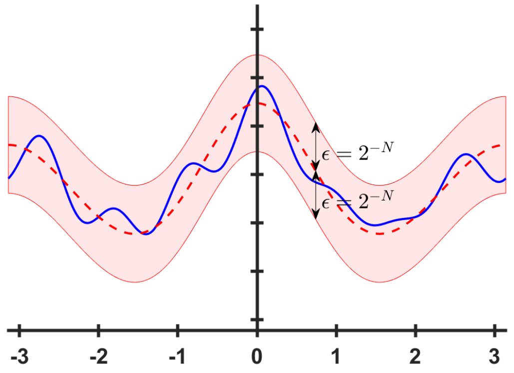
Answer: $\phi' \in L^p_\mathrm{c}(\partial\mathbb{D})$ with $p > 1$.

Theorem:
Let $\phi \in \mathscr{C}_\mathrm{c}(\mathbb{T})$ be strictly positive on $\partial\mathbb{D}$ so that there exists an $p \in \mathbb{R}_\mathrm{c}$, $p > 1$ such that $\phi' \in L^p_\mathrm{c}(\partial\mathbb{D})$.
Then ϕ_+ is a computable continuous function, i.e. $\phi_+ \in \mathscr{C}_\mathrm{c}(\partial\mathbb{D})$.

Theorem:
There exists a strictly positive spectral density $\phi \in \mathscr{C}_\mathrm{c}(\partial\mathbb{D})$ with $\phi' \in L^1_\mathrm{c}(\partial\mathbb{D})$ so that $\phi_+(1) \notin \mathbb{R}_\mathrm{c}$.

Summary

- ▷ There is no closed form expression for the Hilbert transform $\mathrm{H}f$ or the spectral factor ϕ_+.
  $\Longrightarrow$ Numerically approximation methods (on digital computers) are applied to determine $\mathrm{H}f$ or ϕ_+.



- ▷ Numerically approximation:
  Given f or ϕ and $\varepsilon > 0$, determine (in finite time) a confidence interval of width 2ε in which the (unknown) $\widetilde{f}$ or ϕ_+, respectively, lies. $\Rightarrow$ $\widetilde{f}$ or ϕ_+ is computable.

- ▷ Negative results:
  - There exist computable continuous functions f with very good analytic properties (finite energy, absolute continuous, etc.) for $\widetilde{f} = \mathrm{H}f$ is not computable.
  - There exist computable spectral densities ϕ with very decent analytic properties (finite energy, absolute continuous, etc.) for which the spectral factor ϕ_+ is not computable.

- ▷ Positive results:
  - Sharp characterization of sets of functions f and spectral densities ϕ such that $\widetilde{f}$ and ϕ_+ is computable.