Low Mutual and Average Coherence Dictionary Learning Using Convex Approximation

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Dictionary learning

- Dictionary learning problem as follows[1]:

1) Sparse representation:

$$\boldsymbol{X}^{(k+1)} = \underset{\boldsymbol{X}\in\chi}{\operatorname{argmin}} \left\| \boldsymbol{Y} - \boldsymbol{D}^{(k)}\boldsymbol{X} \right\|_F^2 \qquad (2)$$

2) Dictionary update:

$$\boldsymbol{D}^{(k+1)} = \underset{\boldsymbol{D}\in\xi}{\operatorname{argmin}} \left\| \boldsymbol{Y} - \boldsymbol{D}\boldsymbol{X}^{(k+1)} \right\|_F^2 \qquad (3)$$

Stage 1 is an ordinary sparse coding problem [2]-[5], which can be done, for example, by Orthogonal Matching Pursuit (OMP) [6].

Mutual and Average Coherence

- 1) Mutual coherence:

$$\mu = \max_{i \neq j} \frac{|d_i^T d_j|}{\|d_i\|_2 \|d_j\|_2} \text{ [7]}, \; D \in \mathbb{R}^{m \times n} \rightarrow \mu_{welch} = \sqrt{\frac{n-m}{m(n-1)}}$$

$$\mu_{welch} \leq \mu \leq 1 \text{ [8]}$$

- 2) Average coherence:

$$\mu_{avg} = \sqrt{\frac{\|D^T D - I\|_F^2}{n(n-1)}}$$

- Mutual coherence plays an important role in sparse approximation. A signal with a sparse representation x with sparsity level (number of non-zero elements) s, can be recovered from $y = Dx$ when [9]:

$$s \leq \frac{1}{2}\left(1 + \frac{1}{\mu}\right)$$

- According to above equation, dictionaries with low mutual coherence are better for high s.

Decreasing mutual and average coherence

- Some of algorithms reducing mutual coherence of dictionary by solving the following problem [10]:

$$min_{D\in\zeta, X\in\chi, H\in\upsilon}\|Y - DX\|_F^2 + \frac{\lambda}{2}\|D^T D - H\|_F^2 \quad (H)$$

$$\upsilon = \{H \in \mathbb{R}^{n\times n}: H = H^T, h_{ii} = 1, \forall i \max_{i\neq j}|h_{ij}| \leq \mu_0\}$$

This problem does not yield closed form solution for updating the dictionary because the gradient of above cost function over *D* is not linear.

Our proposed method

- Convex approximation[11]:

$$\boldsymbol{D} = \boldsymbol{D}_0 + \boldsymbol{D}\text{-}\, \boldsymbol{D}_0 \quad , \quad \boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{X}\text{-}\, \boldsymbol{X}_0$$

$$\boldsymbol{D}\boldsymbol{X} = (\boldsymbol{D}_0 + \boldsymbol{D} - \boldsymbol{D}_0)(\boldsymbol{X}_0 + \boldsymbol{X} - \boldsymbol{X}_0) \approx \boldsymbol{D_0}\boldsymbol{X} + \boldsymbol{D}\boldsymbol{X_0} - \boldsymbol{D_0}\boldsymbol{X_0}$$

$$\boldsymbol{D}^T\boldsymbol{D} = (\boldsymbol{D_0} + \boldsymbol{D} - \boldsymbol{D_0})^T(\boldsymbol{D_0} + \boldsymbol{D} - \boldsymbol{D_0}) \approx \boldsymbol{D_0^T}\boldsymbol{D} + \boldsymbol{D}^T\boldsymbol{D_0} - \boldsymbol{D_0^T}\boldsymbol{D_0}$$

in which, it is assumed that $\|(\boldsymbol{D} - \boldsymbol{D}_0)(\boldsymbol{X} - \boldsymbol{X}_0)\|_F$ and $\|(\boldsymbol{D} -$

$$(\boldsymbol{D}^*, \boldsymbol{X}^*) = \underset{\boldsymbol{D}\in\xi, \boldsymbol{X}\in\chi}{\operatorname{argmin}} \|\boldsymbol{Y} + \boldsymbol{D_0}\boldsymbol{X_0} - \boldsymbol{D_0}\boldsymbol{X} - \boldsymbol{D}\boldsymbol{X_0}\|_F^2 + \frac{\lambda}{2}\left\|\boldsymbol{D}^T\boldsymbol{D_0} + \boldsymbol{D_0^T}\boldsymbol{D} - \boldsymbol{D_0^T}\boldsymbol{D_0} - \boldsymbol{H}\right\|_F^2$$

Our proposed method *(continued)*

1) Sparse representation:

$$\mathbf{D} = \mathbf{D}_k, \mathbf{D_0} = \mathbf{D}_{k-1}, \mathbf{X_0} = \mathbf{X}_k, \mathbf{Z}_k = \mathbf{Y} - (\mathbf{D}_k - \mathbf{D}_{k-1})\mathbf{X}_k$$

$$\mathbf{X}_{k+1} = \underset{\mathbf{X} \in \chi}{\operatorname{argmin}} \|\mathbf{Z}_k - \mathbf{D}_{k-1}\mathbf{X}\|_{\mathrm{F}}^2$$

2) Dictionary updating:

$$\mathbf{X} = \mathbf{X_0} = \mathbf{X}_{k+1}, \mathbf{D_0} = \mathbf{D}_k$$

$$\boldsymbol{D}_{k+1} = \underset{\boldsymbol{D} \in \xi}{argmin} \|\boldsymbol{Y} - \boldsymbol{D}\boldsymbol{X}_{k+1}\|_F^2 + \frac{\lambda}{2} \left\|\boldsymbol{D}^T\boldsymbol{D}_k + \boldsymbol{D}_k^T\boldsymbol{D} - \boldsymbol{D}_k^T\boldsymbol{D}_k - \boldsymbol{H}_k\right\|_F^2$$

Our proposed method *(continued)*

- Dictionary updating:

$$M(\boldsymbol{D}) = \|\boldsymbol{Y} - \boldsymbol{D}\boldsymbol{X}_{k+1}\|_F^2 + \frac{\lambda}{2}\left\|\boldsymbol{D}^T\boldsymbol{D}_k + \boldsymbol{D}_k^T\boldsymbol{D} - \boldsymbol{D}_k^T\boldsymbol{D}_k - \boldsymbol{H}_k\right\|_F^2$$

$$\nabla_{\boldsymbol{D}} M(\boldsymbol{D}) = (\boldsymbol{D}\boldsymbol{X}_{k+1} - \boldsymbol{Y})\boldsymbol{X}_{k+1}^T + \boldsymbol{\lambda}\boldsymbol{D}_k(\boldsymbol{D}^T\boldsymbol{D}_k + \boldsymbol{D}_k^T\boldsymbol{D} - \boldsymbol{D}_k^T\boldsymbol{D}_k - \boldsymbol{H}_k)$$

$$\boldsymbol{D}\boldsymbol{W}_k + \lambda\boldsymbol{D}_k\boldsymbol{D}^T\boldsymbol{D}_k + \lambda\boldsymbol{D}_k\boldsymbol{D}_k^T\boldsymbol{D} = \boldsymbol{C}_k$$

After some calculation (the details are in the paper), we can write:

$$vec(\boldsymbol{D}_{k+1}) = \left(\boldsymbol{W}_{\boldsymbol{k}} \otimes I_m + I_n \otimes \lambda\boldsymbol{D}_k^T\boldsymbol{D}_k + \lambda\boldsymbol{B}_k\right)^{-1} vec(\boldsymbol{C}_k)$$

Summary of our proposed method(Convex-GSD)

1) Sparse coding:

$$\mathbf{Z}_k = \mathbf{Y} - (\mathbf{D}_k - \mathbf{D}_{k-1})\mathbf{X}_k \Longrightarrow \boldsymbol{X}_{k+1} = \mathrm{OMP}(\boldsymbol{Z}_k, \boldsymbol{D}_{k-1}, s)$$

2) Dictionary updating:

$$vec(\boldsymbol{D}_{k+1}) = \left(\boldsymbol{W}_k \otimes I_m + I_n \otimes \lambda \boldsymbol{D}_k^T \boldsymbol{D}_k + \lambda \boldsymbol{B}_k\right)^{-1} vec(\boldsymbol{C}_k)$$

3) Updating matrix **H** [5]:

$$\mu_0 \geq \mu_{welch} \ and \ \eta = \mathbf{D}_{k+1}^T \mathbf{D}_{k+1}$$

$$\mathbf{H}_{k+1} = \begin{cases} 1 & i = j \\ \eta & i \neq j, |\eta| \leq \mu_0 \\ sgn(\eta)\mu_0 & i \neq j, |\eta| \geq \mu_0 \end{cases}$$

Simulation Results:

- Apply our innovation on the GSD[10] and RAMC[12] and results in Convex-GSD and Convex-RAMC, respectively.
- The criteria to evaluate:

1) Root Mean Square Error (RMSE) :

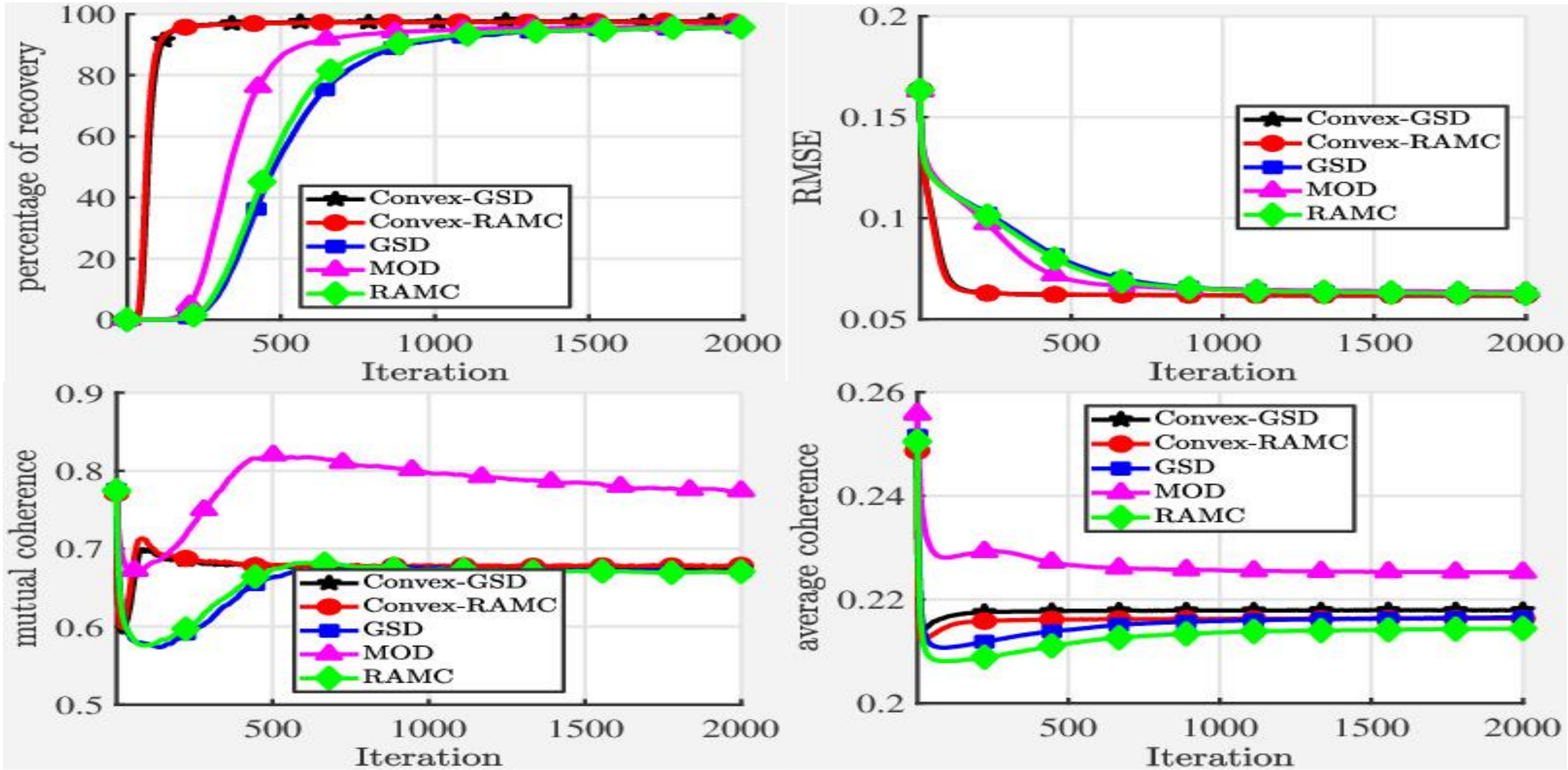
$$\varepsilon_K = \frac{\|\boldsymbol{Y} - \boldsymbol{D}^k \boldsymbol{X}^k\|_F}{\sqrt{mL}}$$

2) Percentage of atom recovery:

$$min_j(1 - |\boldsymbol{D}(:,i)^T \boldsymbol{D_t}(:,j)|) < 0.01.$$

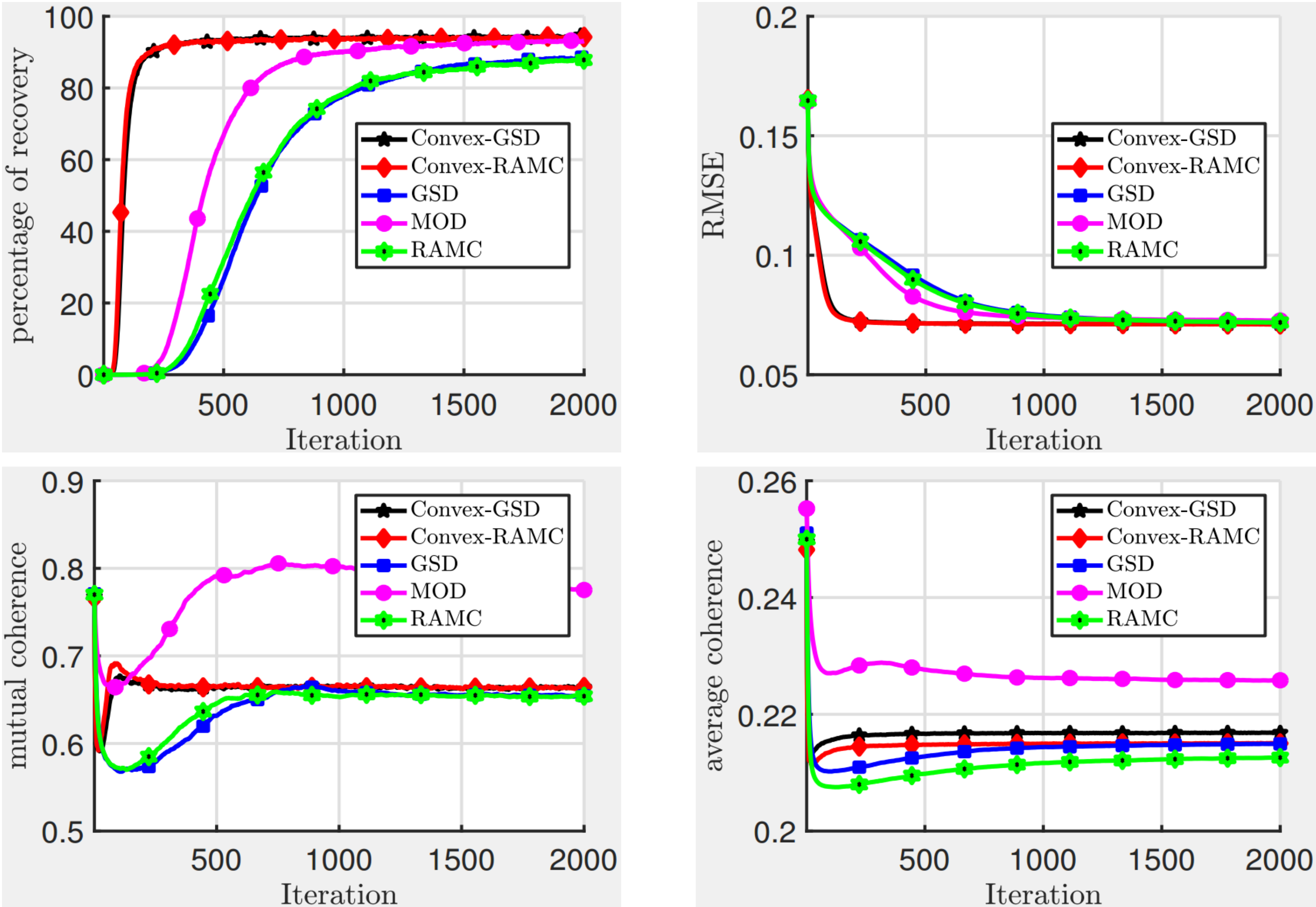
3) Mutual and average coherence.

$\boldsymbol{D} \in \mathbb{R}^{20\times 50}, s = 7, \lambda = 5, SNR = 30\text{dB}, \boldsymbol{Y} \in \mathbb{R}^{20\times 2500}$



According to this slide and slide 12 too, our two algorithms (include Convex) have higher convergence rate and lower RMSE in comparison to the other algorithms while mutual and average coherence are same.

$\boldsymbol{D} \in \mathbb{R}^{20\times 50}, s = 7, \lambda = 5, SNR = 20\text{dB}, \boldsymbol{Y} \in \mathbb{R}^{20\times 2500}$



Number of iterations and average running time (in seconds) for achieving percentage of recovery= 80. Average running times are reported in parentheses. In this table, $s = 7$ and SNR= 30dB are supposed.

Algorithm	$\lambda = 5$	$\lambda = 10$
Convex-GSD	91 (8.3s)	123 (12.1s)
Convex-RAMC	83 (7.5s)	102 (10.1s)
GSD	697 (18.6s)	1913 (59.3s)
RAMC	643 (17.2s)	626 (19.4s)
MOD	452 (9.4s)	449 (10.5s)

According to this table, our algorithms (include Convex) have higher convergence rate in comparison to the other algorithms.

Conclusions

- Convex method for dictionary learning with low mutual coherence.
- Our approach increases the convergence rate and decreases RMSE.
- Mutual and average coherence of our algorithms are reduced well.

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Thank you for your attention