Upscaling Vector Approximate Message Passing

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The model

Consider the recovery of a random signal **x** from a set of linear measurements

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}$$

Where

- $\mathbf{x} \in \mathbb{R}^N$
- $\mathbf{y} \in \mathbb{R}^M$
- $\mathbf{w} \sim N(0, v_w \mathbf{I}_M)$
- $\mathbf{A} \in \mathbb{R}^{M \times N}$

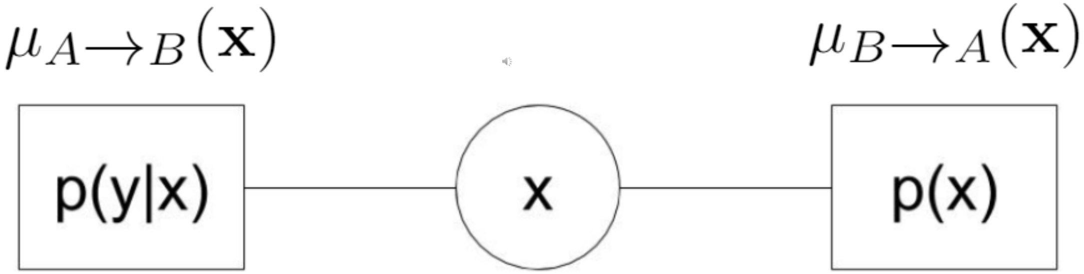
And we consider the compressed sensing scenario $M << N$ with both of a similar order

Inference via Bayes-motivated approach: EP

Assume we can form the posterior for **x** given measurements **y**

$$p(\mathbf{x}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{A}, \mathbf{x})p(\mathbf{x})$$

which can be represented with a factor graph (FG)



On this FG, we employ EP with isotropic Gaussian approximations:

- p(y|x) is approximated by $\mu_{A\rightarrow B}(\mathbf{x}) = N(\mathbf{x}; \mathbf{x}_{A\rightarrow B}, v_{A\rightarrow B}\mathbf{I}_N)$
- p(x) is approximated by $\mu_{B\rightarrow A}(\mathbf{x}) = N(\mathbf{x}; \mathbf{x}_{B\rightarrow A}, v_{B\rightarrow A}\mathbf{I}_N)$

Expectation Propagation updates

On that simple factor graph, the EP update rules are

$$\mu_{A\to B}(\mathbf{x}) = \frac{\mu_A(\mathbf{x})}{\mu_{B\to A}(\mathbf{x})} = \frac{proj\left[\mu_{B\to A}(\mathbf{x})p(\mathbf{y}|\mathbf{x})\right]}{\mu_{B\to A}(\mathbf{x})}$$

$$\mu_{B\to A}(\mathbf{x}) = \frac{\mu_A(\mathbf{x})}{\mu_{A\to B}(\mathbf{x})} = \frac{proj\left[\mu_{A\to B}(\mathbf{x})p(\mathbf{x})\right]}{\mu_{A\to B}(\mathbf{x})}$$

Where the proj[ ] operator is the KL projection on the family of Gaussian distributions with isotropic covariance matrices

Note that the updates are carried out only in terms of moments: **the mean and the variance**.

$$\mu_{A\to B}(\mathbf{x}) = N(\mathbf{x}; \mathbf{x}_{A\to B}, v_{A\to B}\mathbf{I}_N)$$

$$\mu_{B\to A}(\mathbf{x}) = N(\mathbf{x}; \mathbf{x}_{B\to A}, v_{B\to A}\mathbf{I}_N)$$

EP-based algorithm

If one derives these update rules, one can get the following algorithm

Initialization: $\mathbf{x}^0_{B\to A} = 0, \tilde{v}^0_{B\to A} = \tilde{v}_x, t = 0$

while $t < T_{max}$ *and* $\tilde{v}^t_{B\to A} \geq \epsilon$ **do**

- **Block A**
  - $\boldsymbol{\mu}^t_A = \mathbf{g}_A(\mathbf{x}^t_{B\to A}, \tilde{v}_{B\to A}, \tilde{v}_w)$
  - $\frac{1}{\gamma^t_A} = \frac{1}{N}\nabla_{(\mathbf{x}^t_{B\to A})} \cdot \left(\mathbf{A}^T \mathbf{g}_A(\mathbf{x}^t_{B\to A}, \tilde{v}^t_{B\to A}, \tilde{v}_w)\right)$
  - $\mathbf{x}^t_{A\to B} = \mathbf{x}^t_{B\to A} - \gamma^t_A \mathbf{A}^T \boldsymbol{\mu}^t_A$
  - $\tilde{v}^t_{A\to B} = f_A(\boldsymbol{\mu}^t_A, \mathbf{x}^t_{B\to A}, \tilde{v}_{B\to A}, \tilde{v}_w)$

  } computation of $\mu_{A\to B}(\mathbf{x})$

  $N(\mathbf{x}; \mathbf{x}_{A\to B}, v_{A\to B}\mathbf{I}_N)$

- **Block B**
  - $\boldsymbol{\mu}^{t+1}_B = \mathbf{g}_B(\mathbf{x}^t_{A\to B}, \tilde{v}^t_{A\to B})$
  - $\gamma^{t+1}_B = \frac{1}{N}\nabla_{\mathbf{x}^t_{A\to B}} \cdot \mathbf{g}_B(\mathbf{x}^t_{A\to B}, \tilde{v}^t_{A\to B})$
  - $\mathbf{x}^{t+1}_{B\to A} = \frac{1}{1-\gamma^t_B}\left(\boldsymbol{\mu}^{t+1}_B - \gamma^{t+1}_B \mathbf{x}^t_{A\to B}\right)$
  - $\tilde{v}^{t+1}_{B\to A} = f_B(\boldsymbol{\mu}^{t+1}_B, \mathbf{x}^t_{B\to A}, \tilde{v}_{A\to B})$

  } computation of $\mu_{B\to A}(\mathbf{x})$

  $N(\mathbf{x}; \mathbf{x}_{B\to A}, v_{B\to A}\mathbf{I}_N)$

- $t = t + 1$

Output: $\boldsymbol{\mu}^t_B$

Other works

An equivalent form of EP, called Vector Approximate Message Passing (VAMP), was first proposed by Rangan *et al* [1]. Shortly after a similar result was presented by Takeuchi [2].

Both of these works studied the dynamics of EP for the considered problem under the assumption that in the SVD of

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

the singular vector matrix **V** is **Haar distributed**, while **U** and **S** can be any.

Implementation of Block B

Block B

$$\boldsymbol{\mu}_B^{t+1} = \mathbf{g}_B(\mathbf{x}_{A\to B}^t, \tilde{v}_{A\to B}^t)$$
$$\gamma_B^{t+1} = \frac{1}{N}\nabla_{\mathbf{x}_{A\to B}^t} \cdot \mathbf{g}_B(\mathbf{x}_{A\to B}^t, \tilde{v}_{A\to B}^t)$$
$$\mathbf{x}_{B\to A}^{t+1} = \frac{1}{1-\gamma_B^t}\left(\boldsymbol{\mu}_B^{t+1} - \gamma_B^{t+1}\mathbf{x}_{A\to B}^t\right)$$
$$\tilde{v}_{B\to A}^{t+1} = f_B(\boldsymbol{\mu}_B^{t+1}, \mathbf{x}_{B\to A}^t, \tilde{v}_{A\to B})$$

Thus g_B acts as a denoiser with measurements $\mathbf{x}_{A\to B}^t$

The scalar γ_B^{t+1} is the divergence of the denoiser

The function f_B produces an estimate of the MSE $\frac{1}{N}||\mathbf{x}_{B\to A}^{t+1} - \mathbf{x}||^2$

These components were well studied in [4], [5], [6], [7]

Properties of Block A

Block A

$$\left|\begin{array}{l} \boldsymbol{\mu}_A^t = \mathbf{g}_A(\mathbf{x}_{B\to A}^t, \tilde{v}_{B\to A}, \tilde{v}_w) \\ \frac{1}{\gamma_A^t} = \frac{1}{N}\nabla_{(\mathbf{x}_{B\to A}^t)} \cdot \left(\mathbf{A}^T\mathbf{g}_A(\mathbf{x}_{B\to A}^t, \tilde{v}_{B\to A}^t, \tilde{v}_w)\right) \\ \mathbf{x}_{A\to B}^t = \mathbf{x}_{B\to A}^t - \gamma_A^t \mathbf{A}^T \boldsymbol{\mu}_A^t \\ \tilde{v}_{A\to B}^t = f_A(\boldsymbol{\mu}_A^t, \mathbf{x}_{B\to A}^t, \tilde{v}_{B\to A}, \tilde{v}_w) \end{array}\right.$$

It was shown that that the mean $\mathbf{x}_{B\to A}^t$ of the approximated density $\mu_{B\to A}(\mathbf{x})$ is equal to

$$\mathbf{x}_{B\to A}^t = \mathbf{x} + \mathbf{q}_t$$

The function g_A is the LMMSE estimator $\mathbf{g}_A(\mathbf{x}_{B\to A}^t) = \mathbf{W}_t^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x}_{B\to A}^t)$

$$\mathbf{W}_t = \tilde{v}_w \mathbf{I}_M + \tilde{v}_{B\to A}^t \mathbf{A}\mathbf{A}^T$$

- Directly compute the inverse – very slow
- Use SVD – requires storing large matrices; intractable amount of memory

The scalar $\frac{1}{\gamma_A^t}$ is the divergence of $\mathbf{A}^T\mathbf{g}_A$

- The same problems as with g_A

The Block A is intractable when the dimensions of the system are large as in many imaging problems. Alternatives?

Conjugate Gradient (CG) approximation

Use **a few iterations** of CG to approximate the LMMSE

$$\mathbf{g}_A(\mathbf{x}^t_{B\to A}) = \mathbf{W}_t^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x}^t_{B\to A}) = \mathbf{z}_t$$

What about the divergence of the resulting $\mathbf{A}^T\mathbf{g}_A$ and the MSE $\tilde{v}^t_{A\to B}$?

Takeuchi and Wen shown [3] that under Haar **V** this divergence can be estimated for *i* iterations of CG if one has access to *2i + 2* moments of the singular spectrum of **S**

What if we don't have the access to those moments?

The divergence of CG

In [3] it was shown that shown that as $N \to \infty$ and with Haar **V**, the CG function becomes a linear mapping

$$\mathbf{g}_A^{i[t]} = \mathbf{U}\mathbf{H}_t^{i[t]}\mathbf{U}^T$$

of the vector $\mathbf{z}_t$ and the diagonal matrix $\mathbf{H}_t^{i[t]}$ is a function of **S**, v_w and $v_{B\to A}^t$ only.

The from the definition of $\gamma_A^{t,i[t]}$ we can show that

$$\frac{1}{\gamma_A^t} = Tr\left\{\mathbf{H}_t^{i[t]}\mathbf{S}\mathbf{S}^T\right\} = \frac{\frac{1}{N}\mathbf{q}_t^T\mathbf{A}^T\mathbf{g}_A^{i[t]}(\mathbf{z}_t)}{\tilde{v}_{B\to A}^t}$$

which is independent of a particular realization of **w** and $\mathbf{q}_t$ but is only a function of its statistics

Estimating the divergence of CG

Since the divergence is independent of a particular realization of **w** and $\mathbf{q}_t$ but is only a function of its statistics, synthesize

$$\dot{\mathbf{z}}_t = \dot{\mathbf{w}} - \mathbf{A}\dot{\mathbf{q}}_t$$

with

$$\dot{\mathbf{w}} \sim \mathbf{N}(\mathbf{0}, \mathbf{v_w I_M})$$
$$\dot{\mathbf{q}}_\mathbf{t} \sim \mathbf{N}(\mathbf{0}, \mathbf{v^t_{B\to A} I_N})$$

Execute CG on the synthesized data. We expect

$$\frac{1}{\dot{\gamma}_A^{t,i[t]}} = \frac{\frac{1}{N}\dot{\mathbf{q}}_t^T \mathbf{A}^T \mathbf{g}_A^{i[t]}(\dot{\mathbf{z}}_t)}{\tilde{v}^t_{B\to A}}$$

to be close to the result with the real data. Use $\dot{\gamma}_A^{t,i[t]}$ as an estimate of $\gamma_A^{t,i[t]}$

Efficient estimator of MSE $v_{B\to A}^t$

We still need to compute the MSE $\tilde{v}_{A\to B}^t = \frac{1}{N}||\mathbf{x}_{A\to B}^t - \mathbf{x}||^2$

Using the definition of $\mathbf{x}_{A\to B}^t$ and of $\dot{\gamma}_A^{t,i[t]}$, one can show that is it equal to

$$\tilde{v}_{A\to B}^t = (N)^{-1}(\dot{\gamma}_A^{t,i[t]})^2(\boldsymbol{\mu}_A^{t,i[t]})^T\mathbf{A}\mathbf{A}^T\boldsymbol{\mu}_A^{t,i[t]} - v_{B\to A}^t$$

All the components are available

State Evolution of CG-VAMP

It can be shown that the exact solution to

$$\mathbf{g}_A(\mathbf{x}^t_{B\to A}) = \mathbf{W}_t^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x}^t_{B\to A})$$

gives the optimal performance of VAMP w.r.t. the choice of $\mathbf{g}_A(\mathbf{x}^t_{B\to A})$

When we use CG, we sacrifice both **convergence rate** and **the quality of the fixed point** of VAMP

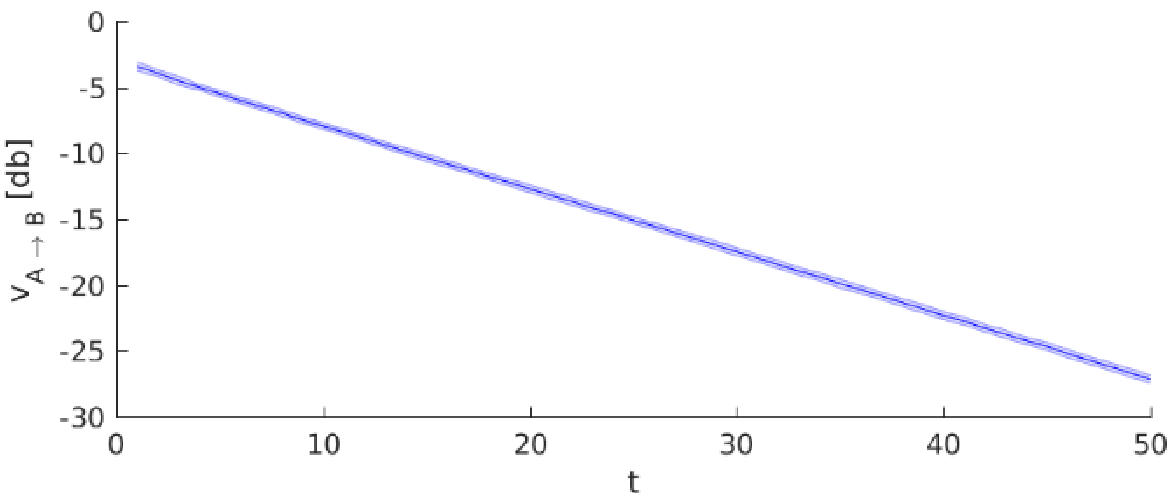
In order to preserve the efficiency and the quality, we adaptively choose the number of CG iterations and iterate while

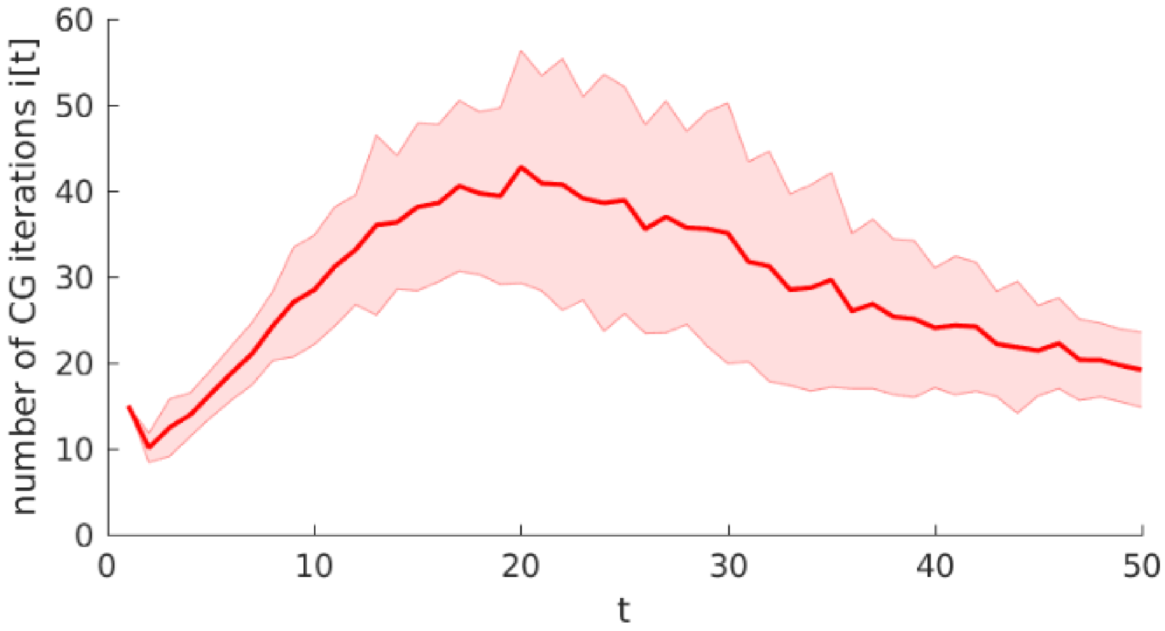
$$\tilde{v}^t_{A\to B}(i) \leq c\tilde{v}^{t-1}_{A\to B}$$

for some constant *c<1* that is larger than for the exact $\mathbf{g}_A(\mathbf{x}^t_{B\to A})$

Simulation results of adaptive CG for VAMP

- x is Bernoulli-Gaussian signal
- $N = 2^{14}$, $M = 2^{13}$
- geometric singular values
- condition number 10 000
- SNR = 40dB
- constant *c=0.9* for the variance reduction





Conclusions

- This work has presented efficient on-the-fly estimation of the variance and divergence terms for CG-VAMP using the concept of a synthetic statistical system
- This implementation does not rely on any prior information about the singular values of **A**
- We have presented an adaptive implementation of CG-VAMP in order to ensure a good convergence rate
- Simulations (not shown) based on Fast ill-conditioned Johnson-Lindenstrauss operators result in both fast and accurate reconstruction

References

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