CONVOLUTIONAL GROUP-SPARSE CODING AND SOURCE LOCALIZATION

Pol del Aguila Pla and Joakim Jaldén, [poldap,jalden]@kth.se

Department of Information Science and Engineering, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology

Non-negative convolutional coding (NN-CC), Blind spatially-variant deconvolution, and Source localization (SL)

Consider the convolutional coding model

$$s = \sum_{k=1}^{K} h_k \circledast x_k + w\,, \tag{1}$$

with $\{x_k\}_1^K \subset \mathbb{T}(M,N)$ some feature maps and $\{h_k\}_1^K$ a dictionary of convolutional kernels with $\|h_k\|_1$ constant across k.

Consider a spatially variant blur, i.e. a model in which

$$s[m,n] = \sum_{\hat{m},\hat{n}=1}^{M,N} y\left[\hat{m},\hat{n}\right] h_{\hat{m},\hat{n}}\left[m-\hat{m}, n-\hat{n}\right] + w[m,n] \tag{2}$$

where $y \in \mathbb{T}_+(M,N)$ is an image and $h_{m,n}$ a spatially variant point-spread function (PSF) of an optical system with $\|h_{m,n}\|_1$ constant across (m,n).

If the feature maps are non-negative $\{x_k\}_1^K \subset \mathbb{T}_+(M,N)$ and $\forall m,n,\ h_{m,n} \in \operatorname{conv}\left(\{h_k\}_1^K\right)$, the convolutional coding model (1) is equivalent to that of a spatially-variant convolution (2). Indeed, if $y = \sum_{k=1}^K x_k$ and $\alpha_k[m,n] = \frac{x_k[m,n]}{y[m,n]}$ if $y[m,n] > 0$, then (1) can be expressed as

$$s[m,n] = \sum_{\hat{m},\hat{n}} y\left[\hat{m},\hat{n}\right] \sum_{k=1}^{K} \alpha_k\left[\hat{m},\hat{n}\right] h_k\left[m-\hat{m}, n-\hat{n}\right] + w[m,n]\,. \tag{3}$$

In diverse scientific scenarios, image data is explained in terms of a number of point- or extended-sources emitting some measurable signal. SL methods automate the accurate localization of these sources. In particular, if we consider a number Q of point-sources that produce signals with shapes $\{h_q\}_1^Q$ with $\|h_q\|_1 = 1$ and scale $y_q > 0$, the image data s follows the model

$$s[m,n] = \sum_{q=1}^{Q} y_q h_q[m-m_q, n-n_q] + w[m,n] = \sum_{q=1}^{Q} (h_q[\hat{m},\hat{n}] \circledast y_q\, \delta[\hat{m}-m_q, \hat{n}-n_q])[m,n] + w[m,n]\,. \tag{4}$$

With the spatially-variant convolutional model in (3), we aim to recover

$$y[m,n] = \sum_{q=1}^{Q} y_q\, \delta[m-m_q, n-n_q], \text{ and } \alpha_k\left[m,n\right] \text{ such that } h_q = \sum_{k=1}^{K} \alpha_k\left[m_q,n_q\right] h_k\,, \tag{5}$$

i.e., the locations of each source, their scale, and a characteritzation of their shapes.

Convolutional sparse coding (CSC) for feature extraction, Background

Under the model (1) and with a given convolutional dictionary $\{h_k\}_1^K$, CSC is often addressed by the convolutional basis pursuit denoising (BPDN) problem

$$\min_{\{x_k \in \mathbb{T}(M,N)\}_1^K} \left\{ \left\| \sum_{k=1}^{K} h_k \circledast x_k - s \right\|_2^2 + \lambda \sum_{k=1}^{K} \|x_k\|_1 \right\}. \tag{6}$$

With respect to standard sparse coding schemes, (6) naturally allows for invariance constraints (e.g., spatial, rotational) to be part of the feature extraction process, and greatly reduces the number of model parameters.

[1] Pol del Aguila Pla and Joakim Jaldén, "Cell detection by functional inverse diffusion and group sparsity – Part I: Theory," 2017, Submitted to *IEEE Transactions on Signal Processing*, available at: `arXiv:1710.01604`.

[2] Pol del Aguila Pla and Joakim Jaldén, "Cell detection by functional inverse diffusion and group sparsity – Part II: Practice," 2017, Submitted to *IEEE Transactions on Signal Processing*, available at: `arXiv:1710.01622`.

Non-negative convolutional group-sparse coding (NN-CGSC)

Based on the coupling between the non-negative x_k in the parallel between (1) and (2), we propose a non-negative group-sparsity regularized version of the BPDN

$$\min_{\{x_k \in \mathbb{T}_+(M,N)\}_1^K} \left\{ \left\| \sum_{k=1}^{K} h_k \circledast x_k - s \right\|_w^2 + \lambda \sum_{g=1}^{G} \sqrt{\sum_{(m,n,k)\in\mathcal{G}_g} x_k^2[m,n]} \right\}.$$

This allows us to force variable selection or variable grouping on the different $\alpha_k[m,n]$s and can be used to

- provide prior information on likely and unlikely combinations of kernels h_k,
- select regions on an image that are likely to be blurred by the same PSF $h_{m,n}$.

Proximal optimization

In order to derive the accelerated proximal gradient (APG) algorithm to solve NN-CGSC, we had to characterize

- the data fidelity penalty, and therefore, the mapping $\{x_k\}_1^K \mapsto \sum_{k=1}^K h_k \circledast x_k$, in terms of its adjoint operator and its norm, which we achieved by generalizing our characteritzation of the diffusion operator in [1].
- the non-negative group sparsity regularizer, in terms of its proximal operator, which we achieved by particularizing our result in [2], where we provided this proximal operator in a functional setting.

In particular, if $R(x)$ is the group-sparsity regularizer and δ_+ the non-negative $(0,\infty)$-indicator we have that

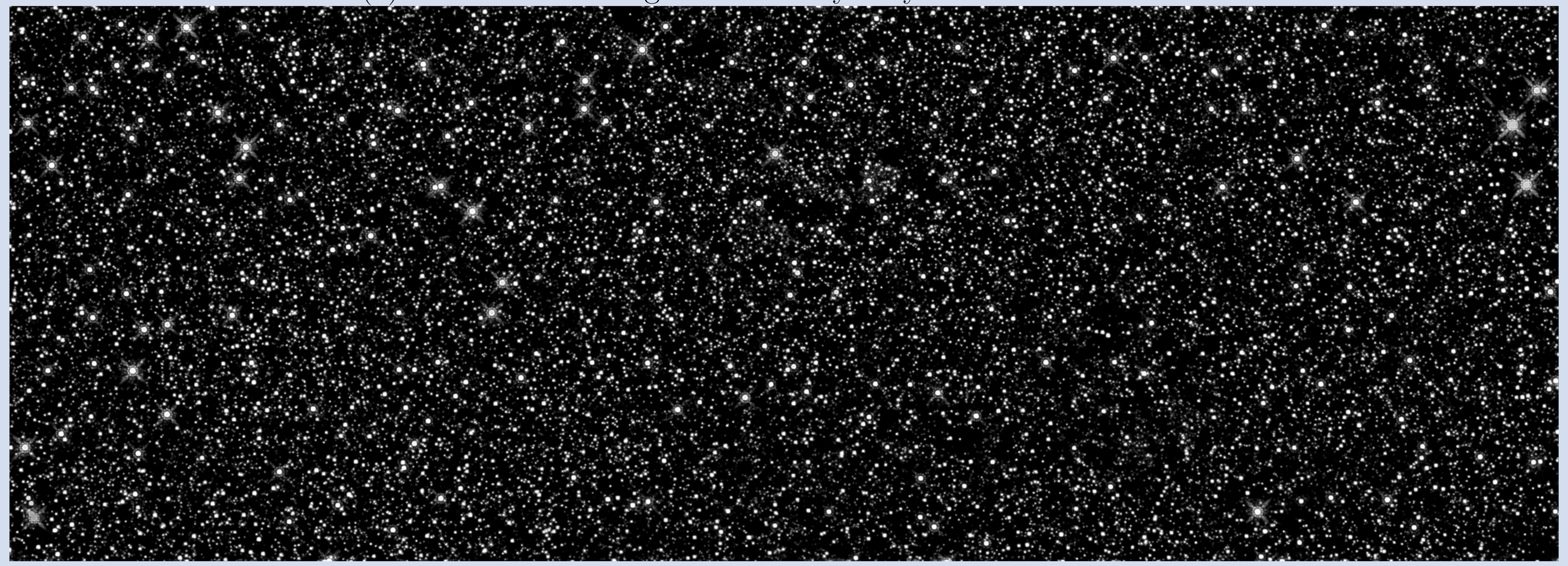
$$\operatorname{prox}_{R+\delta_+} = \operatorname{prox}_R \circ \mathrm{P}_+\,.$$

Accelerated proximal gradient for NN-CGSC

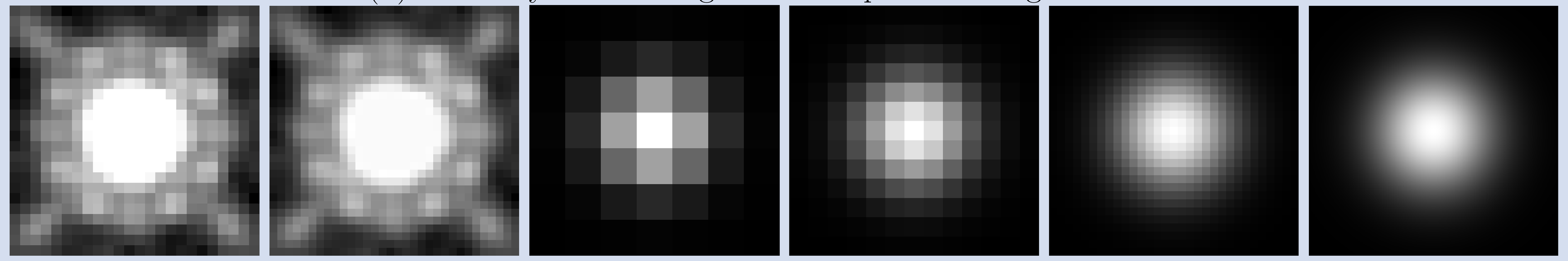
$l \leftarrow 0$
for $k = 1$ **to** K **do**
  $z_k^{(0)} \leftarrow x_k^{(0)}$,
end for
repeat
  $l \leftarrow l + 1$
  $u^{(l)} \leftarrow \sum_{k=1}^{K} h_k \circledast z_k^{(l-1)} - s$
  for $k = 1$ **to** K **do**
    $x_k^{(l)} \leftarrow \left[z_k^{(l-1)} - h_k^{\mathrm{m}} \circledast \left[w \odot u^{(l)}\right]\right]_+$
  end for
  for $g = 1$ **to** G **do**
    $n \leftarrow \sqrt{\sum_{(m,n,k)\in\mathcal{G}_g} \left(x_k^{(l)}[m,n]\right)^2}$
    for $(m,n,k) \in \mathcal{G}_g$ **do**
      $x_k^{(l)}[m,n] \leftarrow \left(1 - \frac{\lambda}{2} n^{-1}\right)_+ x_k^{(l)}[m,n]$
    end for
  end for
  for $k = 1$ **to** K **do**
    $z_k^{(l)} \leftarrow x_k^{(l)} + \alpha(l)\left(x_k^{(l)} - x_k^{(l-1)}\right)$
  end for
until convergence of $\{x_k^{(l)}\}_1^K$

SL by NN-CGSC, a foreground recovery example

(a) Section of an image of the Milky Way's nuclear star cluster



(b) Recovery of the foreground component using NN-CGSC



(c) Left to right, filters $\{h_k\}_1^5$ used for foreground reconstruction $k \in \{1,2,\dots,5\}$ and background removal $k=6$.

Foreground reconstruction using the APG algorithm for NN-CGSC with $M \times N \times 2$ groups of the form $\mathcal{G}_{m,n,p} = \{(m,n,k) : m = i, n = j, k \in \aleph_p\}$, with $p \in \{1,2\}$ and $\aleph_1 = \{1,2\}$ and $\aleph_2 = \{3,4,5\}$. Above, section of a composite color image of the Milky Way's nuclear star cluster, generated by The Hubble Heritage Team, NASA and ESA (STScI-2016-11) from an image capture using Hubble's Wide Field Camera 3. Middle, foreground recovery of the image in grayscale, artificially saturated for printing clarity, showing only lower half of dynamic range.