Conditional Density Driven Grid Design in Point-Mass Filter

Jindřich Duník, **Ondřej Straka**, and Jakub Matoušek

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University of West Bohemia
Pilsen, Czech Republic

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State Estimation

Important for navigation, speech and image processing, fault detection, adaptive and optimal control.

State estimation problem specification:
Consider discrete-time nonlinear stochastic dynamic system with additive noises

$$
\begin{aligned}
\mathrm{x}_{k+1} &= \mathrm{f}_k(\mathrm{x}_k) + \mathrm{w}_k, \qquad & k = 0, 1, 2, \ldots, T \\
\mathrm{z}_k &= \mathrm{h}_k(\mathrm{x}_k) + \mathrm{v}_k, \qquad & k = 0, 1, 2, \ldots, T
\end{aligned}
$$

Given a set of measurements $\mathrm{z}^k \triangleq [\mathrm{z}_0, \mathrm{z}_1, \ldots, \mathrm{z}_k]$ estimate state x_k

Bayesian Recursive Relations (BRRs)

Bayesian approach provides the filtering PDF $p(x_k|z^k)$ obtained by

$$p(x_k|z^k) = \frac{p(x_k|z^{k-1})p(z_k|x_k)}{p(z_k|z^{k-1})}, \qquad \text{(Bayes rule)}$$

$$p(x_k|z^{k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|z^{k-1})dx_{k-1}, \qquad \text{(Chapman-Kolmogorov)}$$

- Analytically tractable for few special cases (linear model with Gaussian noises)
- Approximate solutions (Gaussian assumed filters, particle filter)
- Numerical computation of the BRRs – **Point-Mass filter**

Point-Mass Filter (PMF)

- developed in the seventies,
- based on a numerical solution to the BRRs using *deterministic* grid-based numerical integration rules,
- PMF computes the conditional PDFs at the grid points only,
- selection of the grid points is crucial (affects the PMF accuracy and computational complexity).

Point-Mass Density $\hat{p}(x_k|z^m;\xi_k)$, $m = k-1, k$

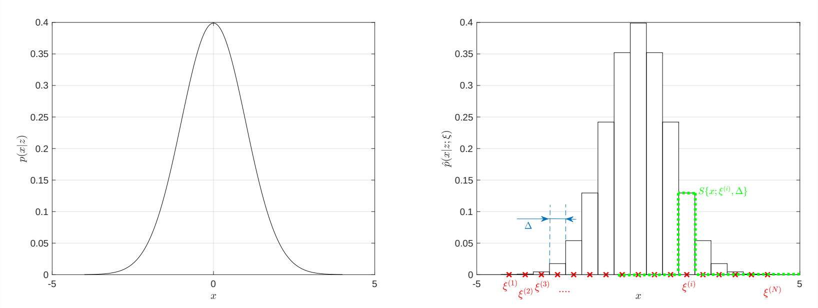
$$\hat{p}(x_k|z^m;\xi_k) \triangleq \sum_{i=1}^{N} P_{k|m}(\xi_k^{(i)}) S\{x_k;\xi_k^{(i)},\Delta_k\},$$

- $P_{k|m}(\xi_k^{(i)}) = p(\xi_k^{(i)}|z^m) / \left(\delta_k \sum_{j=1}^{N} p(\xi_k^{(j)}|z^m)\right)$
- δ_k – volume of the i-th point $\xi_k^{(i)}$ vicinity,
- $\Delta_k = [\Delta_k(1), \ldots, \Delta_k(n_x)]^T$ – a hyper-rectangular vicinity of $\xi_k^{(i)}$
- $S\{x_k;\xi_k^{(i)},\Delta_k\}$ – *selection* function

$$S\{x_k;\xi_k^{(i)},\Delta_k\} = \begin{cases} 1, & \text{if } |x_k(j) - \xi_k^{(i)}(j)| \leq \frac{\Delta_k(j)}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

- $\delta_k = \prod_{i=1}^{n_x} \Delta_k(i) = \int S\{x_k;\xi_k^{(i)},\Delta_k\} dx_k$

Point-Mass Density – Illustration



Point-Mass Filter Algorithm

Initialisation: : Set $k = 0$, construct the initial grid $\{\xi_0^{(i)}\}_{i=0}^N$, define the initial point-mass PDF $\hat{p}(\mathrm{x}_0|\mathrm{z}^{-1};\xi_0)$.

Meas. update: Compute the filtering point-mass PDF $\hat{p}(\mathrm{x}_k|\mathrm{z}^k;\xi_k)$ where
$P_{k|k}(\xi_k^{(i)}) = \frac{p(\mathrm{z}_k|\mathrm{x}_k=\xi_k^{(i)})P_{k|k-1}(\xi_k^{(i)})}{\sum_{i=1}^N p(\mathrm{z}_k|\mathrm{x}_k=\xi_k^{(i)})P_{k|k-1}(\xi_k^{(i)})\delta_k}$.

Grid construction: Construct the new grid $\{\xi_{k+1}^{(j)}\}_{j=0}^N$.

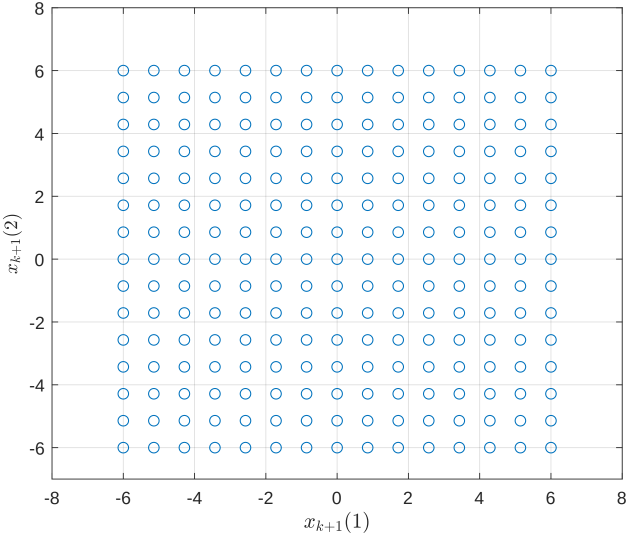
Time update: Compute the predictive point-mass PDF $\hat{p}(\mathrm{x}_{k+1}|\mathrm{z}^k;\xi_{k+1})$ at the new grid points where
$P_{k+1|k}(\xi_{k+1}^{(j)}) = \sum_{i=1}^N p(\xi_{k+1}^{(j)}|\mathrm{x}_k = \xi_k^{(i)})P_{k|k}(\xi_k^{(i)})\delta_k$.

Set $k = k + 1$ and continue with **Meas. update**.

Standard Equidistant Grid (SEG) Design

1.
   - Set number of grid points N
   - Set volume P_{vol} of the conditional PDF to be approximated.
2. Get the mean $\hat{\mathrm{x}}_{k|k}$ and covariance matrix $\mathrm{P}_{k|k}$ of the filtering point-mass PDF $\hat{p}(\mathrm{x}_k|\mathrm{z}^k;\xi_k)$.
3. Compute *approximate* predictive moments $\hat{\mathrm{x}}_{\mathrm{A},k+1|k} \approx \mathrm{E}[\mathrm{x}_{k+1}|\mathrm{z}^k]$, $\mathrm{P}_{\mathrm{A},k+1|k} \approx \mathrm{cov}[\mathrm{x}_{k+1}|\mathrm{z}^k]$.
4. Construct an approximate Gaussian predictive PDF
$$p_{\mathrm{A}}(\mathrm{x}_{k+1}|\mathrm{z}^k) = \mathcal{N}\{\mathrm{x}_{k+1}; \hat{\mathrm{x}}_{\mathrm{A},k+1|k}, \mathrm{P}_{\mathrm{A},k+1|k}\},$$
5. Set the rectangular support $\mathscr{R}$ of the predictive PDF to be approximated by the grid, $\int_{\mathscr{R}} p_{\mathrm{A}}(\mathrm{x}_{k+1}|\mathrm{z}^k)\, d\mathrm{x}_{k+1} \approx P_{\mathrm{vol}}$ ($\mathscr{R}$ is centered at $\hat{\mathrm{x}}_{\mathrm{A},k+1|k}$).
6. Determine the grid points $\{\xi_{k+1}^{(i)}\}_{i=1}^{N}$ equidistantly covering the region $\mathscr{R}$.

SEG Illustration



Goal of the Paper

Standard grid design

(i) The grid design is based on $\hat{p}(\mathsf{x}_{k+1}|\mathsf{z}^k;\xi_{k+1})$ only.

(ii) The support $\mathscr{R}$ is computed assuming a Gaussian PDF.

(iii) The support $\mathscr{R}$ is equidistantly approximated by the points.

New grid design

(i) The design respects $\hat{p}(\mathsf{x}_{k+1}|\mathsf{z}^k;\xi_{k+1})$ and $\hat{p}(\mathsf{x}_{k+1}|\mathsf{z}^{k+1};\xi_{k+1})$.

(ii) The support $\mathscr{R}$ could respect non-Gaussian aspects.

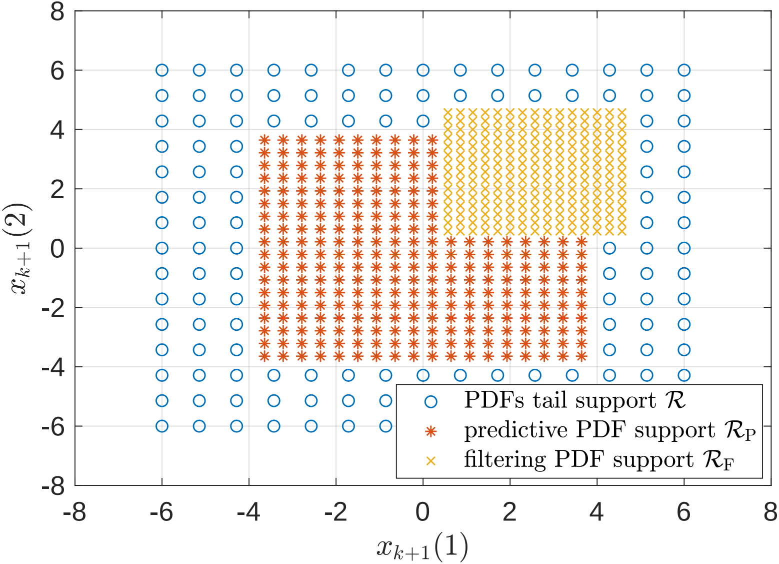
(iii) The part of $\mathscr{R}$ with significant volume of PDFs should be approximated by a denser grid, whereas the remaining support can be approximated by a sparser grid.

Algorithm of conditional density driven grid (CDDG) design

1., 2. Same as in SEG.

3. Compute *approximate* predictive and filtering moments $\hat{x}_{A,k+1|k}$, $P_{A,k+1|k}$, $\hat{x}_{A,k+1|k+1}$, $P_{A,k+1|k+1}$.
   Optionally, compute higher-order moments.
4. Construct *approximate* predictive and filtering Gaussian PDFs $p_A(x_{k+1}|z^k)$ and $p_A(x_{k+1}|z^{k+1})$, respectively.
5. Set rectangular support $\mathscr{R}$
   $\int_{\mathscr{R}} p(x_{k+1}|\ z^k) dx_{k+1} \geq P_{\mathrm{vol}}$ and $\int_{\mathscr{R}} p(x_{k+1}|z^{k+1}) dx_{k+1} \geq P_{\mathrm{vol}}$.
6. Set predictive and filtering sub-regions $\mathscr{R}_P$ and $\mathscr{R}_F$ centered at $\hat{x}_{A,k+1|k}$ and $\hat{x}_{A,k+1|k+1}$ $\int_{\mathscr{R}_P} p(x_{k+1}|z^k) dx_{k+1} \approx c_P P_{\mathrm{vol}}$ and $\int_{\mathscr{R}_F} p(x_{k+1}|z^{k+1})\ dx_{k+1} \approx c_F P_{\mathrm{vol}}$ with $c_F, c_P \in \langle 0.8, 0.95 \rangle$.
   Optionally, use higher-order moments to fine-tune $\mathscr{R}_P$ and $\mathscr{R}_F$.
7. Approximate $\mathscr{R}_P$ and $\mathscr{R}_F$ by a dense grid, and the remaining complement of $\mathscr{R}$ by a sparse grid.

CDDG Illustration



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Numerical Illustration of CDDG-based PMF

Terrain-aided navigation scenario

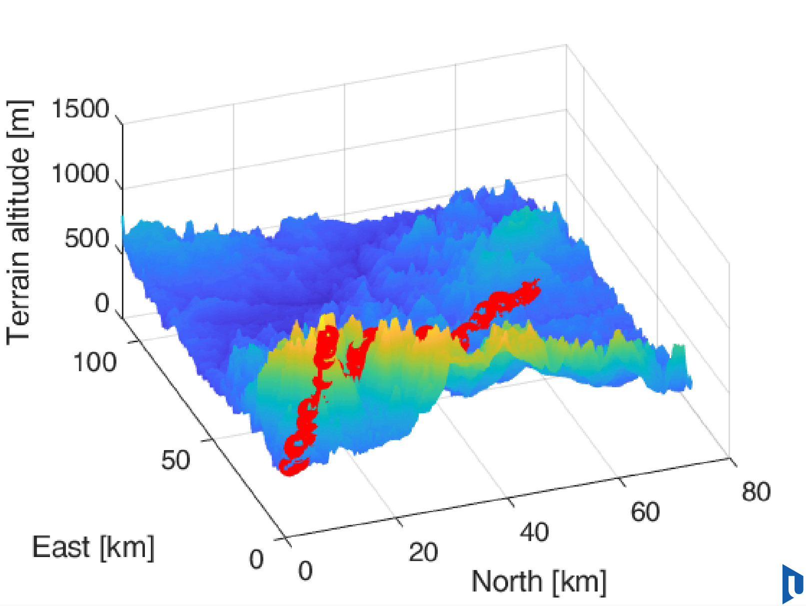
$$\mathrm{x}_{k+1} = \mathrm{x}_k + \mathrm{u}_k$$

- x_k – vehicle horizontal position in north and east directions
- $\mathrm{u}_k = [300, 300]^T$ – shift vector (from INS or odometer)
- w_k is uncertainty in u_k, $p(\mathrm{w}_k) = \mathcal{N}\{\mathrm{w}_k; 0_2, 10^2\mathrm{I}_2\}$

$$z_k = h_k(\mathrm{x}_k) + v_k$$

- z_k – measurement of the terrain altitude below the vehicle (from e.g. barometric altimeter or radar altimeter)
- $h_k(\cdot)$ – terrain map connecting the position and the altitude
- v_k – sensor reading uncertainty and map error, $p(v_k) = \mathcal{N}\{v_k; 0, 8^2\}$

Trajectory Illustration



Compared Point-Mass Filters

- PMF_{TRUE} with high number of grid points $N = 7225$ providing "almost true" state estimate $p(\mathsf{x}_k|\mathsf{z}^k)$,
- PMF_{ST} with the *standard* equidistant allocation with $N = 289$ providing the conditional PDF $\hat{p}_{\text{ST}}(\mathsf{x}_k|\mathsf{z}^k; \xi_k)$,
- PMF_{CDDG} with the *proposed* CDDG with $N = 286$ providing the conditional PDF $\hat{p}_{\text{CDDG}}(\mathsf{x}_k|\mathsf{z}^k; \xi_k)$,

$M = 10^3$ Monte-Carlo simulations using two criteria;

(i) the filtering PDF integral error
$\text{IE} = \frac{1}{T+1} \sum_{k=0}^{T} \int |p(\mathsf{x}_k|\mathsf{z}^k) - \hat{p}(\mathsf{x}_k|\mathsf{z}^k; \xi_k)| d\mathsf{x}_k$

(ii) PMF execution time

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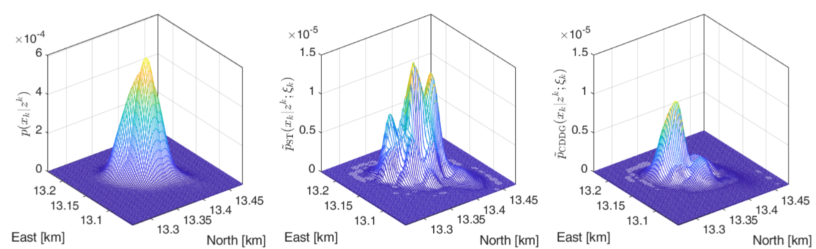
Results

	$\mathrm{PMF}_{\mathrm{TRUE}}$	$\mathrm{PMF}_{\mathrm{ST}}$	$\mathrm{PMF}_{\mathrm{CDDG}}$
IE	–	0.056	**0.024**
time [sec]	72	11	12



Conclusion

- state estimation of nonlinear stochastic dynamic systems by the point-mass filter
- novel conditional density driven grid design
- suitable combination of the dense and sparse grids
- significant PMF state estimation improvement while preserving the number of grid points, and thus the computational complexity

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