Recovery of binary sparse signals from compressed linear measurements via polynomial optimization

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Outline

Introduction

Problem 1

Solve

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \{0,1\}^n, \quad \mathbf{x} \text{ sparse}$$

given

$$\mathbf{y} \in \mathbb{R}^m, \; \mathbf{A} \in \mathbb{R}^{m,n}, \quad m < n$$

Applications

- digital communications
- digital image processing
- source separation
- localization
- operations research, e.g., knapsack problem

Introduction

Problem 1

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \{0,1\}^n, \quad \mathbf{x} \text{ sparse }, \quad \mathbf{y} \in \mathbb{R}^m, \ \mathbf{A} \in \mathbb{R}^{m,n}, \quad m < n$$

Properties

$$\left.\begin{array}{l}\text{Compressed sensing problem} \\ \text{Boolean (0/1) optimization problem}\end{array}\right\} \Rightarrow \textit{Binary compressed sensing}$$

Literature

- A. Flinth and G. Kutyniok, "**PROMP**: A sparse recovery approach to lattice-valued signals", Appl. Comput. Harmon. Anal., 2018
- S. Fosson, "Non-convex approach to binary compressed sensing", Asilomar Conf., 2018 (**RWR**: ℓ_1-reweighting algorithm for local minimization)

Polynomial optimization approach

Problem 1: binary compressed sensing

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \{0,1\}^n, \quad \mathbf{x} \text{ sparse }, \quad \mathbf{y} \in \mathbb{R}^m, \ \mathbf{A} \in \mathbb{R}^{m,n}, \quad m < n$$

Problem 2: polynomial optimization (POP)

$$\min_{\mathbf{x} \in [0,1]^n} \sum_{i=1}^{n} \left(\mathbf{x}_i - \mathbf{x}_i^2\right) \quad \text{s. t.} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

Proposition

Let us assume that the solution $\mathbf{z} \in \{0,1\}^n$ to Problem 1 is unique. Then, $\mathbf{z}$ is the unique solution to Problem 2, for any $m \geq 1$.

How to solve Problem 2?

Problem 2

$$\min_{\mathbf{x}\in[0,1]^n} \sum_{i=1}^{n} \left(\mathbf{x}_i - \mathbf{x}_i^2\right) \quad \text{s. t.} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

- Problem 2 is non-convex
- Lasserre's polynomial theory: the global minimum can be found by solving a hierarchy of semidefinite programs (SDPs) $\rightarrow$ convex

Proposition

The global minimum of the first order SDP (= Shor's relaxation) of Problem 2 is the global minimum of Problem 2. The global minimizer can be extracted if the solution of the SDP has rank 1.

Complexity reduction via chordal sparsity

- $\mathbf{x}$ n-dimensional $\rightarrow$ SDP n^2-dimensional
- Chordal sparsity: if the graph that represents the sparsity pattern of the SDP matrices is chordal, then the SDP can be decomposed into smaller sub-problems $\Rightarrow$ reduces the complexity
- Chordal sparsity $\leftrightarrow$ running intersection property

Y. Zheng et al. "Chordal decomposition in operator-splitting methods for sparse semidefinite programs", Math. Progamm., 2019 $\rightarrow$ **CDCS:** Cone decomposition conic solver (Matlab)

$$\min_{\mathbf{x}\in[0,1]^n, \theta\in\mathbb{R}^{m,n-1}} \sum_{i=1}^{n} \left(\mathbf{x}_i - \mathbf{x}_i^2\right) \text{ s. t., for each } i = 1\dots, m,$$

$$\begin{aligned} \mathbf{y}_i - \mathbf{A}_{i,1}\mathbf{x}_1 + \theta_{i,1} &= 0 \\ -\mathbf{A}_{i,2}\mathbf{x}_2 - \theta_{i,1} + \theta_{i,2} &= 0 \\ &\vdots \\ -\mathbf{A}_{i,n}\mathbf{x}_n - \theta_{i,n-1} &= 0 \end{aligned}$$

Noisy case

Problem 3: noisy binary compressed sensing

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon, \quad \mathbf{x} \in \{0,1\}^n, \quad \mathbf{y} \in \mathbb{R}^m, \ \mathbf{A} \in \mathbb{R}^{m,n}, \quad m < n$$

Noise $\epsilon \in \mathbb{R}^m$ with $\|\epsilon\|_\infty \leq \eta$

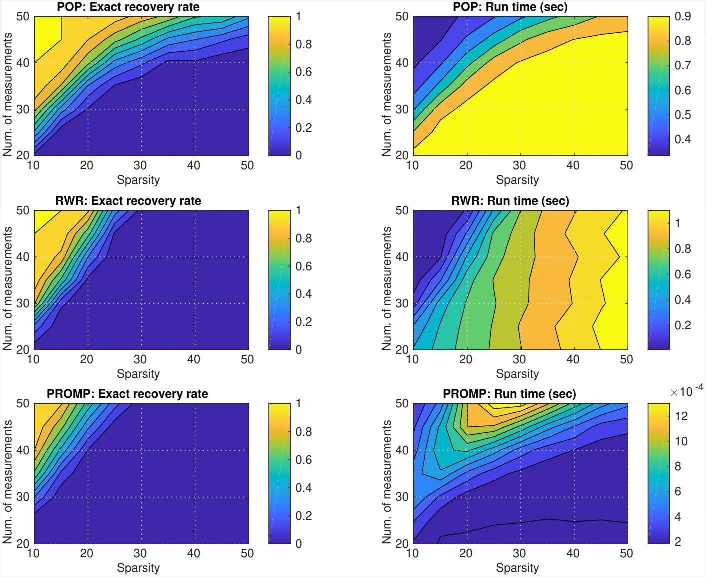
Problem 4: polynomial optimization (POP)

$$\min_{\mathbf{x} \in [0,1]^n} \sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{x}_i^2) \ \text{ s. t. } \ \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_\infty \leq \eta$$

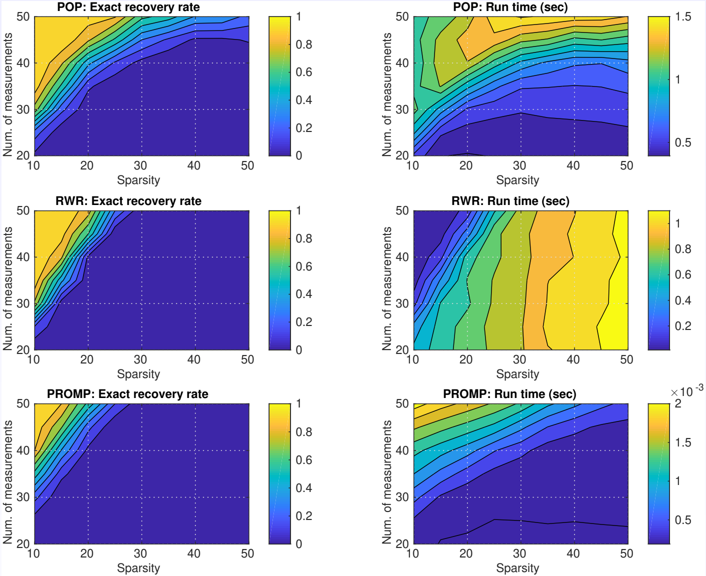
In the noisy case as well we can apply

- SDP relaxation
- Chordal sparsity

Noise-free experiment $n = 100, m \in [20, 50]$



Noisy experiment $n = 100, m \in [20, 50], \eta = 0.05$



Conclusions and future work

Conclusions

- We propose a (non-convex) polynomial approach to binary compressed sensing
- The effectiveness of the proposed method leverages on Lasserre's polynomial optimization theory
- An enhancement is obtained with respect to state-of-the-art binary compressed sensing algorithms

Future work

- further reduction of numerical complexity
- extension to non-binary alphabets
- extension to non-sparse mixed-integer problems, e.g., knapsack problems

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