On the Stability of Polynomial Spectral Graph Filters

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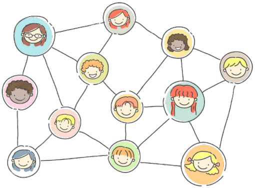
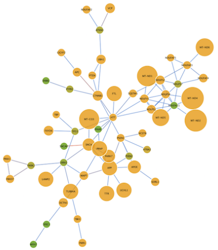
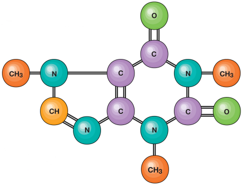
Graph structured data

- Many types of data reside on a graph domain
- Nodes encode an entity
- Edges encode pairwise relationships between entities

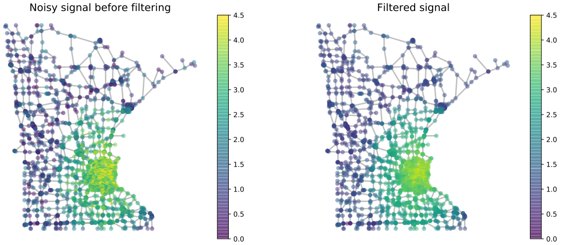






Graph Signal Processing

- We need efficient tools to analyse graph structured data
- Graph signal processing (GSP) generalises signal processing tools, such as filtering, to graph structured data
- A graph signal is a mapping from nodes to a scalar value $x : \mathcal{V} \rightarrow \mathbb{R}$. It can be represented as a vector $\mathbf{x} \in \mathbb{R}^n$.



Motivation

- Topology can be noisy if estimated from data
  - How much impact can this have on using graph filters to analyse the data?
- Understanding of adversarial examples in the graph topology
  - What characterises successful adversarial attacks?

Research questions

- Are spectral graph filters robust to changes in the underlying topology?
- Are spectral graph filters sensitive to certain perturbations of fixed magnitude?
  - If so, can we characterise the perturbations?

Spectral graph theory

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ be a simple graph
- The degree d_i of node i is the number of nodes adjacent to node i
- Degree matrix $\mathbf{D} = \mathrm{diag}(d_1, \ldots d_n)$
- Normalised Laplacian matrix $\mathcal{L}(\mathcal{G}) = \mathbf{I}_n - \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{1/2}$

Spectral filtering

- Laplacian eigendecomposition $\mathcal{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$
  - $\mathbf{\Lambda} = \text{diag}(\lambda_0, \ldots \lambda_{n-1})$ where $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq 2$ are eigenvalues
  - $\mathbf{U}$ is corresponding eigenvector matrix (graph Fourier basis)
- Graph Fourier transform $\hat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$
- Inverse graph Fourier transform $\mathbf{x} = \mathbf{U}\hat{\mathbf{x}}$
- Filtering is given by

$$\mathbf{U}\text{diag}(g(\lambda_1), \ldots g(\lambda_n))\mathbf{U}^T\mathbf{x} = \mathbf{U}g(\mathbf{\Lambda})\mathbf{U}^T\mathbf{x} = g(\mathcal{L})\mathbf{x}$$

Are spectral graph filters robust to changes in the underlying topology?

Polynomial filters

- Our work focus on polynomial filters

$$g_\theta(\mathcal{L})\mathbf{x} = \sum_{k=0}^{K} \theta_k \tilde{\mathcal{L}}^k \mathbf{x}, \quad \tilde{\mathcal{L}} = \mathcal{L} - \mathbf{I}_n.$$

- Scaling is such that eigenvalues lie in $[-1, 1]$.

Polynomial filters are robust

- Consider a perturbation in the underlying topology giving Laplacian matrix $\mathcal{L}_p$ and let the error be $\mathbf{E} = \mathcal{L}_p - \mathcal{L}$
- Output distance is defined as

$$\|g_\theta(\mathcal{L}) - g_\theta(\mathcal{L}_p)\|$$

- For sufficiently small perturbations ($\|\mathbf{E}\|_2 \leq 1$) Levie et al.[1] proved that

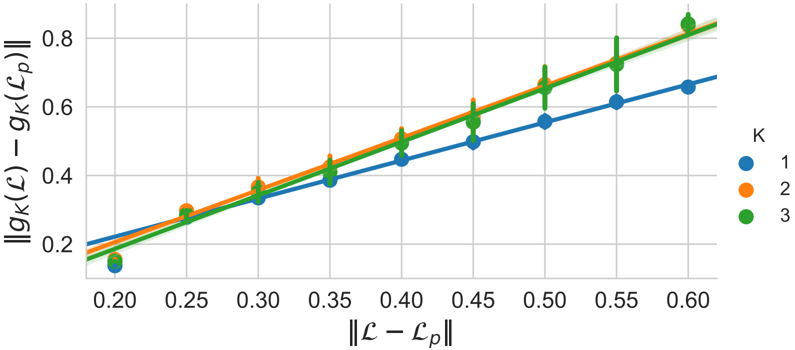
$$\|g_\theta(\mathcal{L}) - g_\theta(\mathcal{L}_p)\| \in \mathcal{O}(\|\mathbf{E}\|)$$

- For order K polynomial filters it holds that

$$\|g_\theta(\mathcal{L}) - g_\theta(\mathcal{L}_p)\| \leq \frac{1}{4}\|\theta\|_1 (K^2 - 1)\left(\frac{K+1}{K-1}\right)^K \|\mathbf{E}\|.$$

[1] R. Levie, E. Isufi, and G. Kutyniok. "On the Transferability of Spectral Graph Filters". In: *arXiv:1901.10524* (2019).

Output distance scales with $\|\mathbf{E}\|_2$



Are spectral graph filters sensitive to certain perturbations of fixed magnitude?

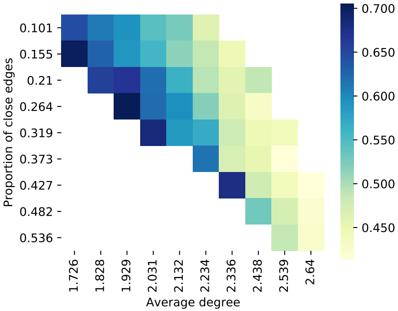
Edge deletion

- Consider removing a fixed number of edges from a graph, what influences the magnitude of $\|\mathbf{E}\|_2$?
  - Entries of the Laplacian matrix depend on degree of end points
  - A change in the Laplacian matrix may be due to multiple edges if they are sufficiently close

Edge deletion

The following empirically lead to larger values of $||\mathbf{E}||_2$:

- Removing edges $e = (u, v)$ with small values of $d_u d_v$ where d_u, d_v is the degree of the endpoints
- Removing edges which are pairwise close to each other



Conclusions and Future Directions

- Change in the output of polynomial graph filters are bounded linearly in $\|\mathbf{E}\|_2$
- Structural properties such as degree distribution appear to give insight into the magnitude of $\|\mathbf{E}\|_2$
- Can we establish bounds on $\|\mathbf{E}\|_2$ which depend on structural properties?
- Some graph neural network architectures use spectral filters for learning representations. Is the output of these graph neural networks also sensitive to certain perturbations?