A Stochastic Model of Block Segmentation Based on the Quadtree and the Bayes Code for It

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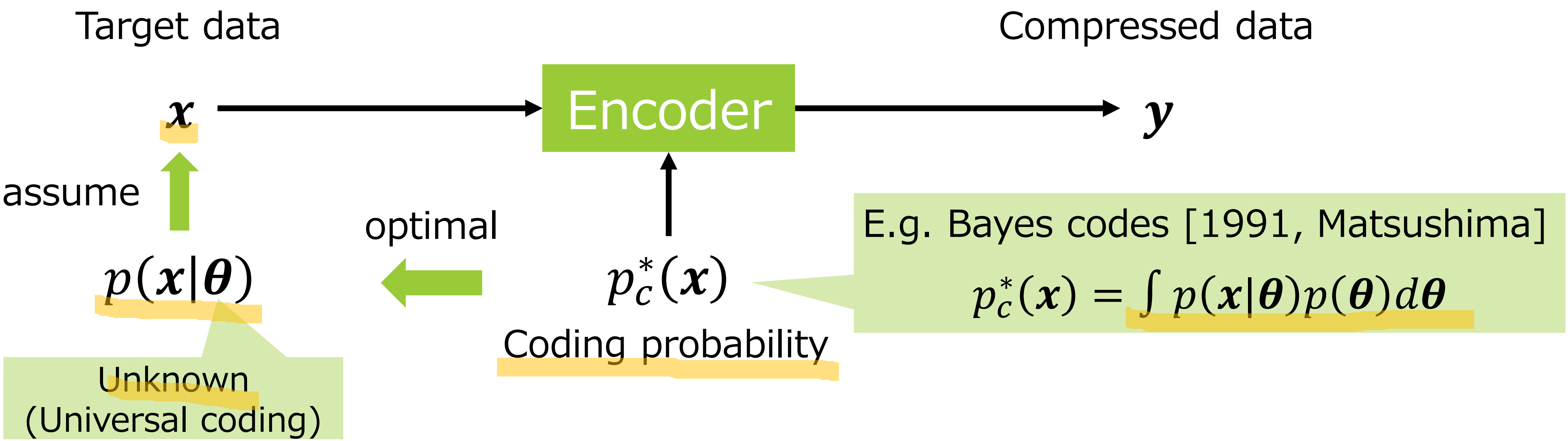
Lossless compression of general data

Information theoretical approach

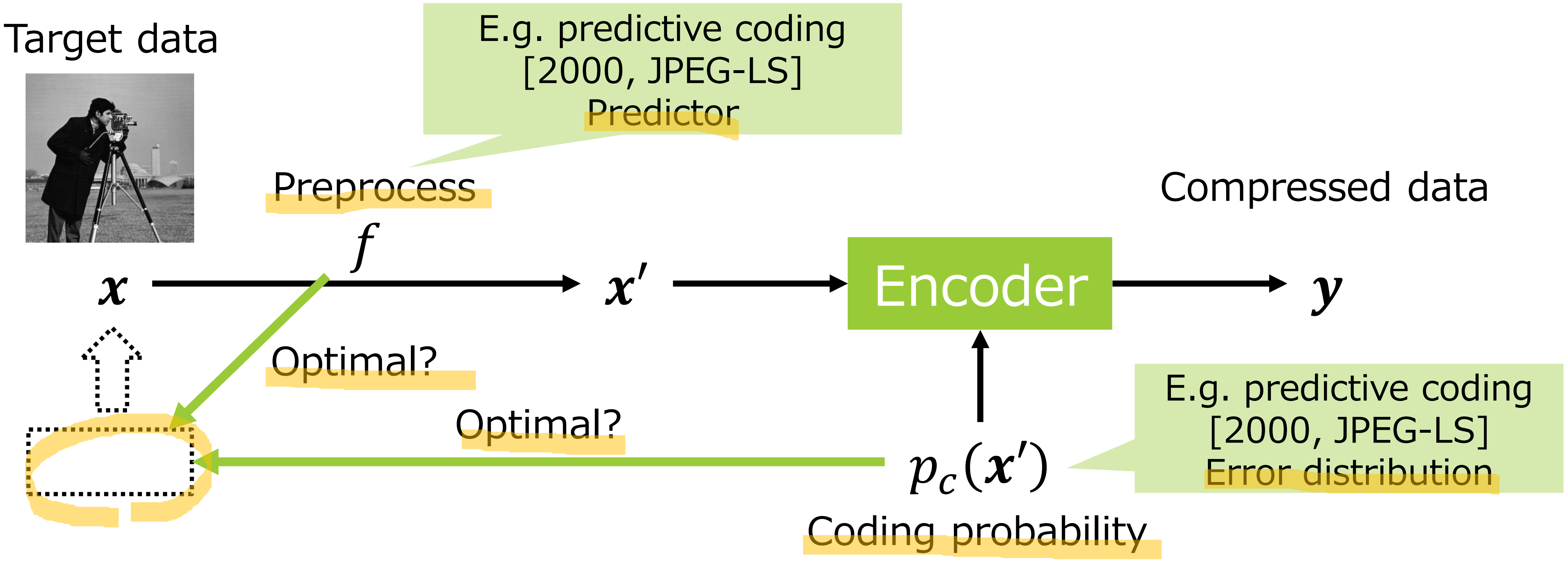


The important factors

- The efficiency of the algorithm to calculate $p_c^*(\boldsymbol{x})$
- The flexibility of the stochastic model $p(\boldsymbol{x}|\boldsymbol{\theta})$

Lossless image compression

- Approaches in most previous studies



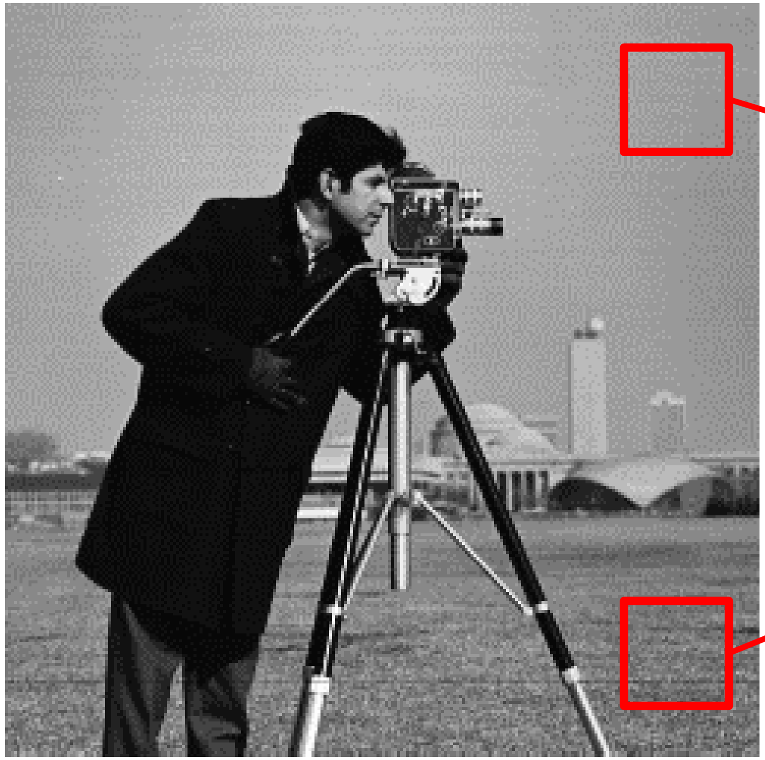
- Difficulty to discuss the optimality of f and $p_c(\boldsymbol{x}')$

Purposes of this study

1. We propose a stochastic model that effectively represents target images.
2. We derive an optimal coding probability.
3. We derive an efficient algorithm to calculate it without loss of the optimality.

Target data

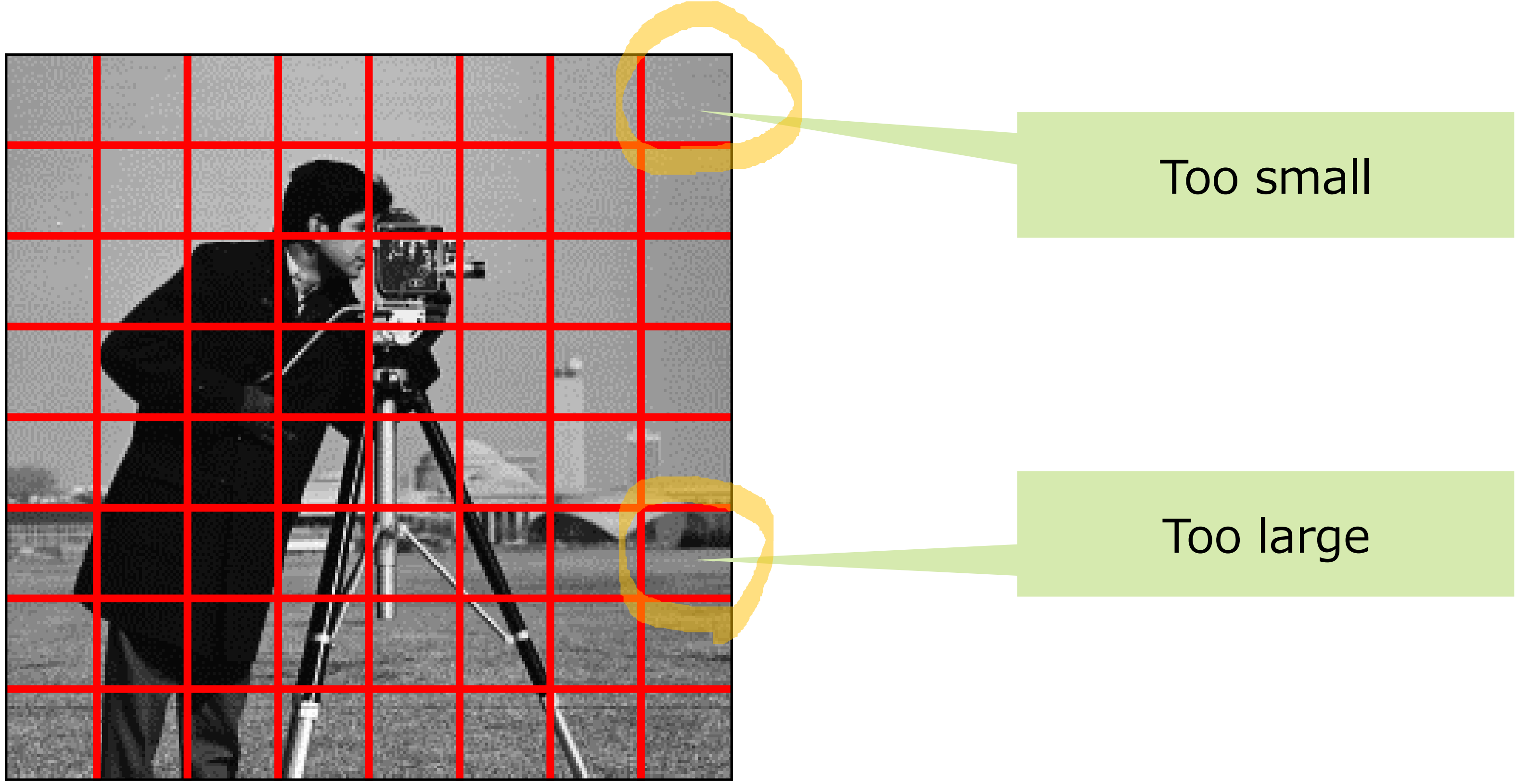
- Images with non-stationarity



$$p(\boldsymbol{x}|\boldsymbol{\theta}) \nparallel p(\boldsymbol{x}|\boldsymbol{\theta}')$$

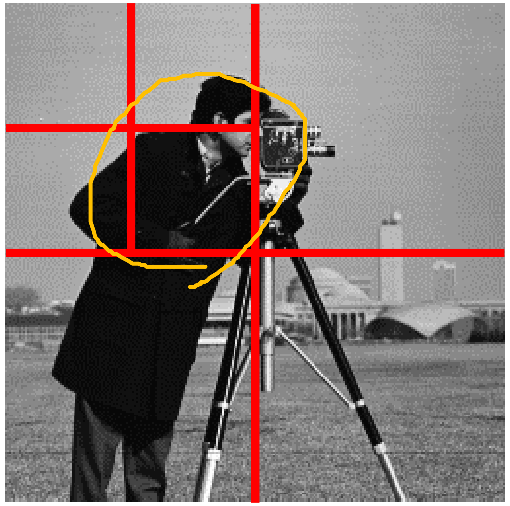
Trivial way

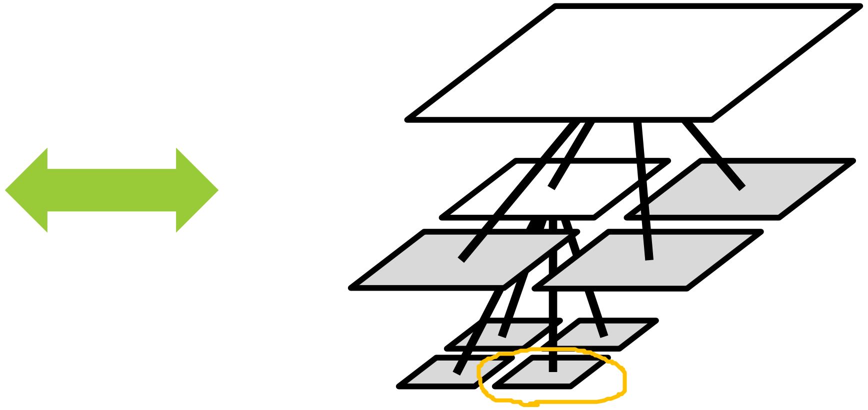
- Divide the image into fixed size blocks
- Assume different stochastic models to them



Quadtree

- It effectively represents variable block size segmentation
-

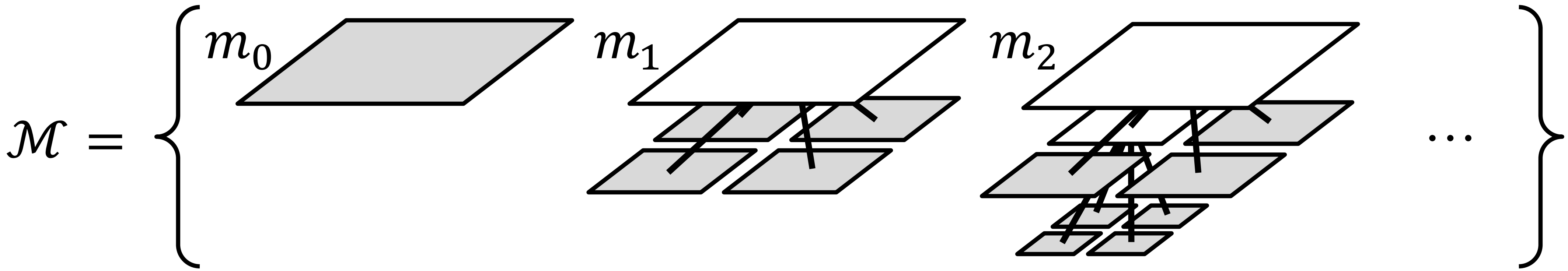




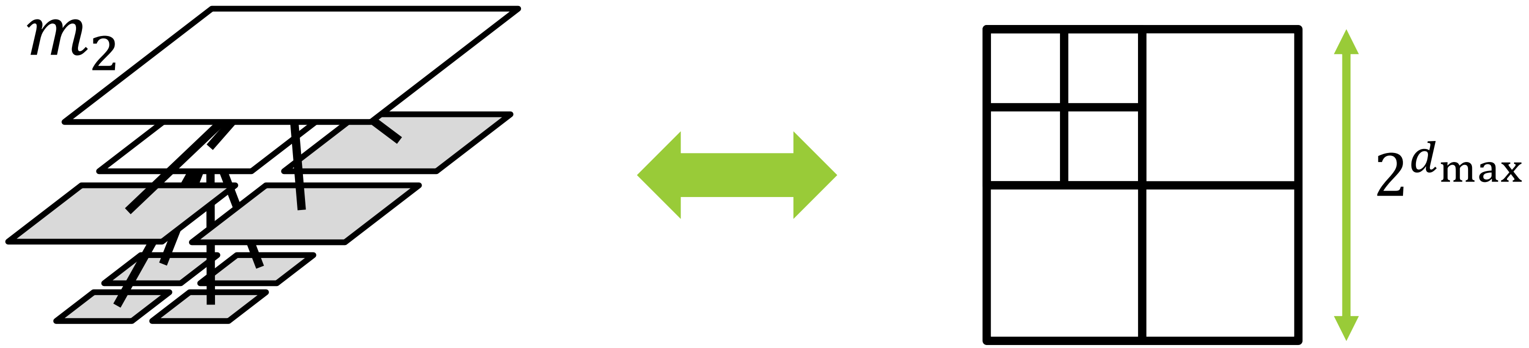
- Previous studies regard it just as a procedure
- In this study, we regard the quadtree as a part of stochastic generative model of the images.

Proposed stochastic model

- Let both of the width and height are $2^{d_{\max}}$.
- Consider the set of the quadtrees whose depth $\leq d_{\max}$.

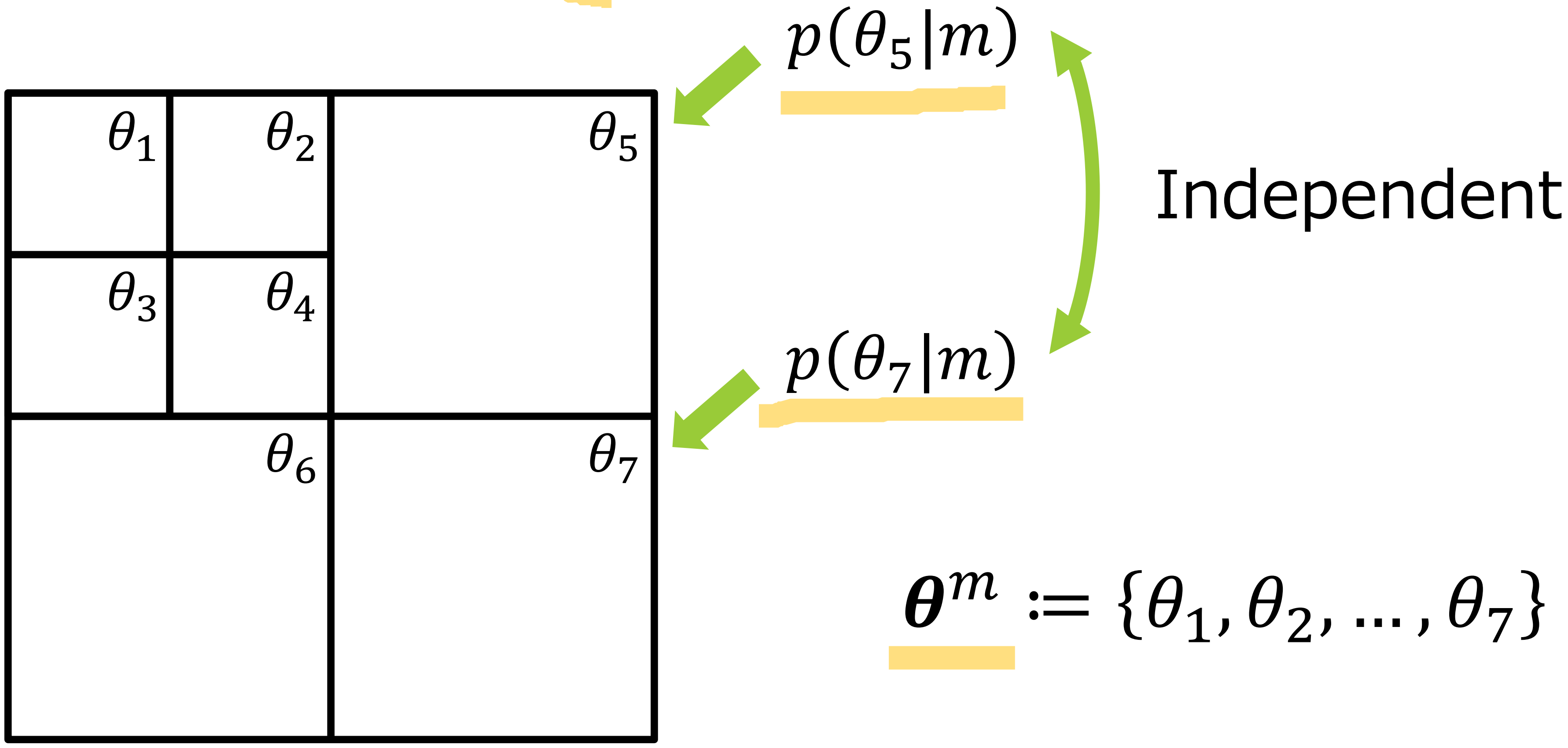


- One of them is chosen with probability $p(m)$.



Proposed stochastic model

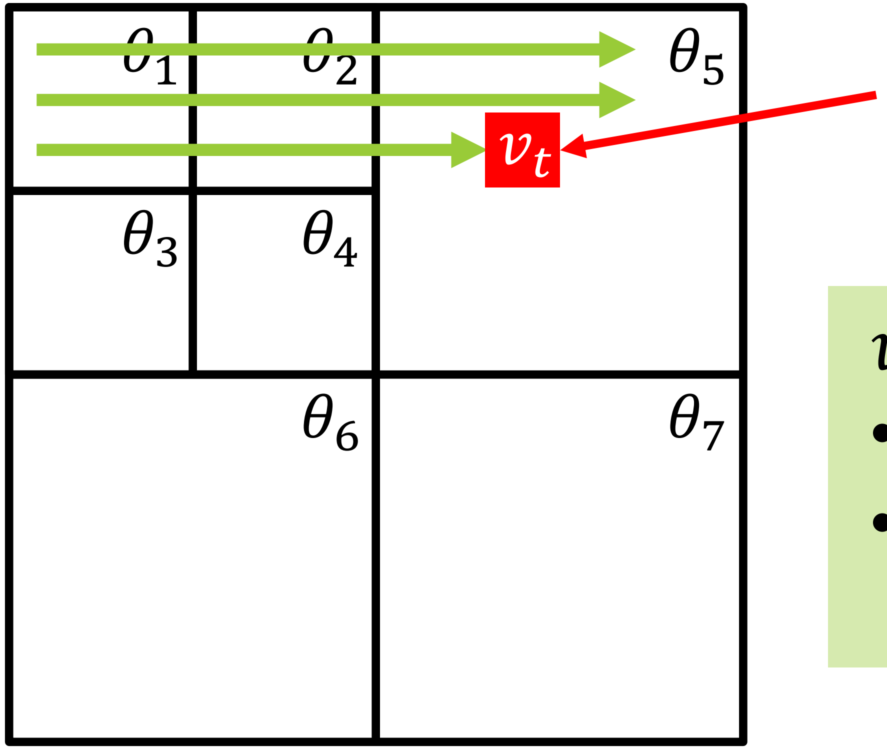
- Parameter θ_s is independently assigned to each block s with probability $p(\theta_s|m)$.



$$\boldsymbol{\theta}^m := \{\theta_1, \theta_2, \dots, \theta_7\}$$

Proposed stochastic model

- Pixel value v_t at block s is generated in order of the raster scan with probability $p(v_t|v^{t-1}, \theta_s, m)$.



$p(v_t|v^{t-1}, \theta_5, m)$

v_t depends only on
- The past sequence v^{t-1}
- The parameter θ_s of the block s which contains v_t

The Bayes code

- We cannot use $p(v_t|v^{t-1},\theta^m,m)$ because true m and $\boldsymbol{\theta}^m$ are unknown.
- We estimate it by $\hat{p}_c(v_t|v^{t-1})$ in Bayesian manner.

- Optimal coding probability $p_c^*(v_t|v^{t-1})$ for our model

$$p_c^*(v_t|v^{t-1})$$
$$=\sum_{m\in\mathcal{M}} p(m|v^{t-1}) \int p(v_t|v^{t-1},\boldsymbol{\theta}^m,m)p(\boldsymbol{\theta}^m|v^{t-1},m)\,\mathrm{d}\boldsymbol{\theta}^m$$

- Three computationally hard parts
  - 1. The summation w.r.t. m ← A recursive structure of quadtree
  - 2. The posterior $p(m|v^{t-1})$ ← Special prior (Detailed in the paper)
  - 3. The integral w.r.t. $\boldsymbol{\theta}^m$ ← Conjugate prior

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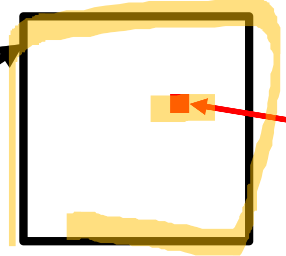
The derived algorithm

- Consider the following case

The whole image



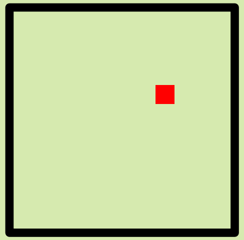
v_t: The pixel we want to compress

- $p_c^*(v_t|v^{t-1})$

$$= (1 - g_{s_\lambda|t-1}) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})$$

Posterior — Coding Probability — Posterior — Coding Probability

The model whose minimal block which contains v_t is root block s_λ



The other models

The derived algorithm

- $p_c^*(v_t|v^{t-1})$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})$$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda)$$

$$+ \underbrace{g_{s_\lambda|t-1}\left(1 - g_{s_{\lambda\,\mathrm{child}}|t-1}\right)}_{\text{Posterior}} \underbrace{q(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}_{\text{Coding Probability}}$$

$$+ \underbrace{g_{s_\lambda|t-1} g_{s_{\lambda\,\mathrm{child}}|t-1}}_{\text{Posterior}} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child\,child}})}_{\text{Coding Probability}}$$

The derived algorithm

- $p_c^*(v_t|v^{t-1})$

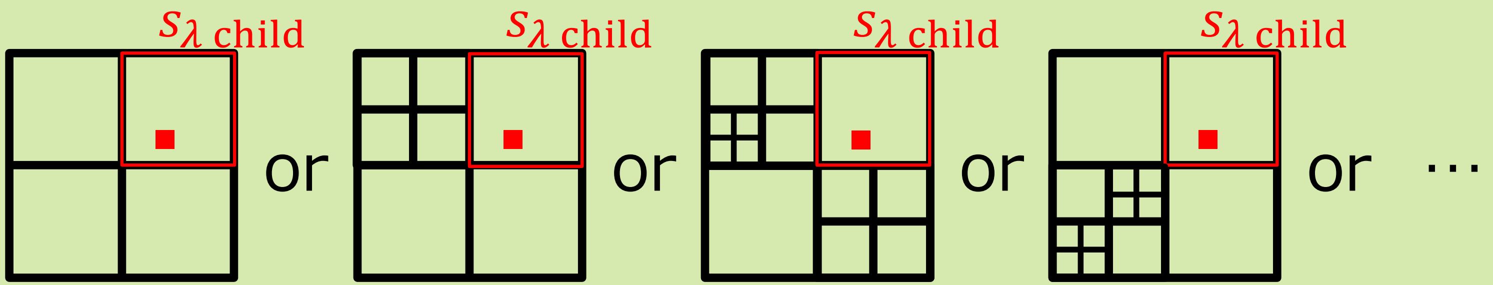
$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} p_c(v_t|v^{t-1}, s_{\lambda\,\text{child}})$$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda)$$

$$+ g_{s_\lambda|t-1}\left(1 - g_{s_{\lambda\,\text{child}}|t-1}\right) q(v_t|v^{t-1}, s_{\lambda\,\text{child}})$$

Posterior   Coding Probability

$$+ \cdots{}_1 g_{s_{\lambda\,\text{child}}|t-1} p_c(v \cdots, s_{\lambda\,\text{child child}})$$



The models whose minimal block which contains v_t is $s_{\lambda\,\text{child}}$

The derived algorithm

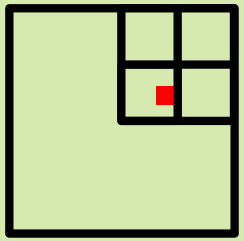
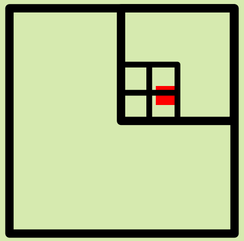
- $p_c^*(v_t|v^{t-1})$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda) + g_{s_\lambda|t-1} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}$$

$$= \left(1 - g_{s_\lambda|t-1}\right) q(v_t|v^{t-1}, s_\lambda)$$

$$+ \underbrace{g_{s_\lambda|t-1}\left(1 - g_{s_{\lambda\,\mathrm{child}}|t-1}\right)}_{\text{Posterior}} \underbrace{q(v_t|v^{t-1}, s_{\lambda\,\mathrm{child}})}_{\text{Coding Probability}}$$

$$+ \underbrace{g_{s_\lambda|t-1} g_{s_{\lambda\,\mathrm{child}}|t-1}}_{\text{Posterior}} \underbrace{p_c(v_t|v^{t-1}, s_{\lambda\,\mathrm{child\,child}})}_{\text{Coding Probability}}$$

The other models:  or  or ⋯

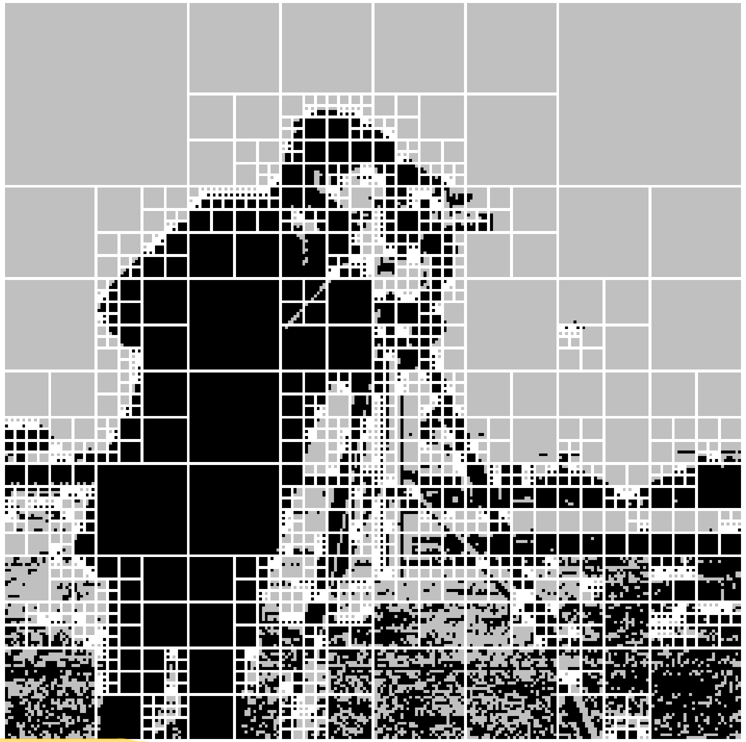
Experiment 1

- Purpose: To confirmation of the Bayes optimality
- Setting:
  - We generated 1000 binary images (64x64) as follows:
    1. Generate m according to $p(m)$ (detailed in the paper).
    2. Generate θ_s according to $Beta(\theta_s|\alpha, \beta)$ for each block s.
    3. Generate v_t according to $Bern(v_t|\ \theta_s)$ for each block s.
  - We compressed them by
    - the proposed method
    - the method with fixed size block segmentation
- Result: Average coding rates (bit/pel)

Proposed	Fixed size 4	Fixed size 8	Fixed size 16
0.619	0.705	0.659	0.679

Experiment 2

- Purpose: To demonstrate the model flexibility
- We compressed the binarized image of the cameraman.
- Result: MAP estimated model $m^{\mathrm{MAP}} = \mathrm{argmax}_{m \in \mathcal{M}} p(m|\boldsymbol{v})$.



Conclusion and future works

- Conclusion
  - We proposed the novel stochastic model with the quadtree.
  - We derived the Bayes optimal coding probability for it.
  - We derived the efficient algorithm to calculate it.

- Future works
  - Our proposed model can be used for other problems
    - Image generation
    - Image inpainting
    - Future extraction