Novel Quaternion Matrix Factorisations

S. Enshaeifar, C. Cheong Took, S. Sanei, and D. P. Mandic*

Faculty of Engineering and Physical Sciences,
University of Surrey, Guildford, U.K.
*Department of Electrical and Electronic Engineering,
Imperial College London, London, U.K.

Outline

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  - Alamouti-based communication problem
- Conclusion

The big picture

Quaternions are of interest in

- Computer Science
- Aeronautics
- Mathematics

Analysis in $\mathbb{C}$ and $\mathbb{H}$

- Analysis of phase information $x = |x|\mathrm{e}^{\phi}$
- Complex-valued problems in communication
- Diagonalisation of covariance matrices: a widely used procedure in signal processing algorithms such as PCA and ICA

Good to know: joint diagonalisation in $\mathbb{C}$

- Advances in $\mathbb{C}$: the necessity of 'augmented statistics' to incorporate the covariance and pseudo-covariance matrices:

$$\mathbf{z} = \mathbf{z}_r + \imath \mathbf{z}_\imath \in \mathbb{C} \;\rightarrow\; \begin{cases} \Re\{\mathbf{z}\} = \frac{1}{2}(\mathbf{z} + \mathbf{z}^*) \\ \Im\{\mathbf{z}\} = \frac{1}{2\imath}(\mathbf{z} - \mathbf{z}^*) \end{cases}$$

$$\mathbf{C}_\mathbf{z} = E[\mathbf{z}\mathbf{z}^H] \qquad \mathbf{P}_\mathbf{z} = E[\mathbf{z}\mathbf{z}^T]$$

$$\mathring{\mathbf{z}} = [\mathbf{z}, \mathbf{z}^*]^T \;\rightarrow\; \mathbf{C}_{\mathring{\mathbf{z}}} = \begin{bmatrix} \mathbf{C}_\mathbf{z} & \mathbf{P}_\mathbf{z} \\ \mathbf{P}^*_\mathbf{z} & \mathbf{C}^*_\mathbf{z} \end{bmatrix}$$

Strong uncorrelating transform (SUT): joint diagonalisation of $\mathbf{C}_\mathbf{z}$ and $\mathbf{P}_\mathbf{z}$ allows for the augmented signal processing algorithms in $\mathbb{C}$, known as SUT.

Useful properties

- **Quaternion involution** $(\cdot)^\eta$: an important notion for $\mathbb{H}$

$$\mathbf{q} = \mathbf{q}_r + \imath\mathbf{q}_\imath + \jmath\mathbf{q}_\jmath + \kappa\mathbf{q}_\kappa \in \mathbb{H} \;\rightarrow\; \begin{cases} \Re\{\mathbf{q}\} = \frac{1}{2}(\mathbf{q} + \mathbf{q}^*) \\ \Im_\eta\{\mathbf{q}\} = \frac{1}{2\eta}(\mathbf{q} - \mathbf{q}^{\eta *}) \end{cases}$$

$$\mathbf{q}^\eta = -\eta\mathbf{q}\eta, \quad \text{e.g.} \quad \mathbf{q}^\imath = -\imath\mathbf{q}\imath = \mathbf{q}_r + \imath\mathbf{q}_\imath - \jmath\mathbf{q}_\jmath - \kappa\mathbf{q}_\kappa$$

- Augmented statistics are also essential for $\mathbb{H}$ to incorporate the covariance and η-covariance matrices:

$$\mathbf{C}_\mathbf{q} = E[\mathbf{q}\mathbf{q}^H] \qquad \mathbf{C}_{\mathbf{q}^\eta} = E[\mathbf{q}\mathbf{q}^{\eta H}]$$

$$\mathring{\mathbf{q}} = [\mathbf{q}, \mathbf{q}^\imath, \mathbf{q}^\jmath, \mathbf{q}^\kappa]^T$$

Quaternion uncorrelating transform (QUT): simultaneous diagonalisation of $\mathbf{C}_\mathbf{q}$ and $\mathbf{C}_{\mathbf{q}^\eta}$ represents quaternion uncorrelating transform (QUT).

Useful properties

(1) A Hermitian matrix: $\mathbf{A} = \mathbf{A}^H \rightarrow$ standard covariance matrix

(2) An η-Hermitian matrix: $\mathbf{A} = \mathbf{A}^{\eta H} \rightarrow$ η-covariance matrices

$$\mathbf{C}_{\mathbf{x}^\alpha} = E\{\mathbf{x}\mathbf{x}^{\alpha H}\} \qquad \alpha \in \{\imath, \jmath, \kappa\} \tag{1}$$

$$\mathbf{x}\mathbf{x}^{\alpha H} = \begin{bmatrix} x_1 {x_1}^{\alpha *} & x_1 {x_2}^{\alpha *} & \cdots & x_1 {x_N}^{\alpha *} \\ x_2 {x_1}^{\alpha *} & x_2 {x_2}^{\alpha *} & \cdots & x_2 {x_N}^{\alpha *} \\ \vdots & \vdots & \ddots & \vdots \\ x_N {x_1}^{\alpha *} & x_N {x_2}^{\alpha *} & \cdots & x_N {x_N}^{\alpha *} \end{bmatrix}$$

Quaternion Takagi factorisation[1] for η-Hermitian matrix $\mathbf{A}$

$$\mathbf{A} = \mathbf{Q}\mathbf{S}_\eta\mathbf{Q}^{\eta H} \begin{cases} \mathbf{Q} \text{ is a quaternion unitary matrix} \\ \mathbf{S}_\eta \text{ is a real-valued non-negative diagonal matrix} \end{cases}$$

[1] MATLAB code available at http://www.surrey.ac.uk/cs/people/clive_cheong_took/index.htm

Simultaneous diagonalisations in $\mathbb{H}$ - case 1

Introduction of η-Hermitian: three sets of joint diagonalisation for matrices $\mathbf{A}, \mathbf{B} \in \mathbb{H}$:

(1) If $\mathbf{A}$ and $\mathbf{B}$ are both Hermitian, there exists a matrix $\mathbf{M} \in \mathbb{H}$ such that $\mathbf{M}^H\mathbf{A}\mathbf{M}$ and $\mathbf{M}^H\mathbf{B}\mathbf{M}$ are both diagonal if and only if $\mathbf{A}\mathbf{B}$ is Hermitian, i.e. $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$

Since $\mathbf{A} = \mathbf{U}\mathbf{S}_a\mathbf{U}^H$ and $\mathbf{B}$ are both Hermitian:

$$\mathbf{D} = \mathbf{U}\mathbf{S}_a{}^{-\frac{1}{2}}\mathbf{U}^H \quad \text{so} \begin{Bmatrix} \mathbf{D}\mathbf{A}\mathbf{D}^H = \mathbf{I} \\ \mathbf{D}\mathbf{B}\mathbf{D}^H = \mathbf{W}\mathbf{\Lambda}_b\mathbf{W}^H \end{Bmatrix}$$

consider $\mathbf{M} = \mathbf{D}^H\mathbf{W}$, thus:

$$\mathbf{M}^H\mathbf{A}\mathbf{M} = \mathbf{W}^H\mathbf{D}\mathbf{A}\mathbf{D}^H\mathbf{W} = \mathbf{I} = \mathbf{\Lambda}_a$$

$$\mathbf{M}^H\mathbf{B}\mathbf{M} = \mathbf{W}^H(\mathbf{D}\mathbf{B}\mathbf{D}^H)\mathbf{W} = \mathbf{\Lambda}_b$$

Simultaneous diagonalisations in $\mathbb{H}$ - case 2

(2) If $\mathbf{A}$ and $\mathbf{B}$ are both η-Hermitian, there exists a unitary matrix $\mathbf{M} \in \mathbb{H}$ such that $\mathbf{M}^{\eta H}\mathbf{A}\mathbf{M}$ and $\mathbf{M}^{\eta H}\mathbf{B}\mathbf{M}$ are both diagonal if and only if $\mathbf{A}\mathbf{B}^{\eta}$ is normal, i.e. $(\mathbf{A}\mathbf{B}^{\eta})(\mathbf{A}\mathbf{B}^{\eta})^{H} = (\mathbf{A}\mathbf{B}^{\eta})^{H}(\mathbf{A}\mathbf{B}^{\eta})$

A single unitary matrix $\mathbf{M}$ is sufficient to diagonalise $\mathbf{A}$ and $\mathbf{B}$ simultaneously.

$\mathbf{A} \overset{\text{Q-SVD}}{=} \mathbf{U}\mathbf{S}\mathbf{V}^{H}$

$\mathbf{A} \overset{\text{Q-Takagi}}{=} \mathbf{Q}\mathbf{S}\mathbf{Q}^{\eta H}$ where $\begin{cases} \mathbf{D} = \mathbf{V}^{\eta H}\mathbf{U} \\ \mathbf{Q} = \mathbf{U}(\mathbf{D}^{\eta})^{\frac{1}{2}} \end{cases}$

Consider $\mathbf{M} = \mathbf{Q}^{\eta}$, thus:

$$\mathbf{M}^{\eta H}\mathbf{A}\mathbf{M} = \mathbf{Q}^{H}\mathbf{A}\mathbf{Q}^{\eta} = \mathbf{Q}^{H}(\mathbf{Q}\mathbf{S}\mathbf{Q}^{\eta H})\mathbf{Q}^{\eta} = \mathbf{S}$$

diagonality of $\mathbf{M}^{\eta H}\mathbf{B}\mathbf{M}$ can be proved using the normality of $\mathbf{A}\mathbf{B}^{\eta}$

Simultaneous diagonalisations in $\mathbb{H}$ - case 3

(3) If $\mathbf{A}$ is Hermitian and $\mathbf{B}$ is η-Hermitian, there exists a matrix $\mathbf{M} \in \mathbb{H}$ such that $\mathbf{M}^H\mathbf{A}\mathbf{M}$ and $\mathbf{M}^{\eta H}\mathbf{B}\mathbf{M}$ are both diagonal if and only if $\mathbf{B}\mathbf{A}$ is η-Hermitian, i.e. $\mathbf{B}\mathbf{A} = (\mathbf{B}\mathbf{A})^{\eta H} = \mathbf{A}^{\eta}\mathbf{B}$

Since $\mathbf{A} = \mathbf{U}\mathbf{S}_a\mathbf{U}^H$ is Hermitian and $\mathbf{B}$ is η-Hermitian:

$$\mathbf{D} = \mathbf{U}{\mathbf{S}_a}^{-\frac{1}{2}}\mathbf{U}^H \quad \text{so} \left\{ \begin{array}{c} \mathbf{D}\mathbf{A}\mathbf{D}^H = \mathbf{I} \\ \mathbf{D}^{\eta}\mathbf{B}(\mathbf{D}^{\eta})^{\eta H} = \mathbf{W}\mathbf{\Lambda}_b\mathbf{W}^{\eta H} \end{array} \right\}$$

Consider $\mathbf{M} = \mathbf{D}^H\mathbf{W}^{\eta}$, thus:

$$\mathbf{M}^H\mathbf{A}\mathbf{M} = \mathbf{W}^{\eta H}\mathbf{D}\mathbf{A}\mathbf{D}^H\mathbf{W}^{\eta} = \mathbf{I} = \mathbf{\Lambda}_a$$

$$\mathbf{M}^{\eta H}\mathbf{B}\mathbf{M} = \mathbf{W}^H(\mathbf{D}^{\eta}\mathbf{B}\mathbf{D}^H)\mathbf{W}^{\eta} = \mathbf{\Lambda}_b$$

Generalisation of case 3: quaternion uncorrelating transform

- **Assumptions in case 3:** $\mathbf{A}$ is Hermitian and $\mathbf{B}$ is η-Hermitian
- **Results of case 3:** $\mathbf{M}^H\mathbf{A}\mathbf{M} = \mathbf{I}$ and $\mathbf{M}^{\eta H}\mathbf{B}\mathbf{M} = \mathbf{\Lambda}_b$
- **Remark:** In general, covariance matrix is Hermitian and η-covariance matrix is η-Hermitian

$$\Downarrow$$

Quaternion uncorrelating transform (QUT): for a random quaternion vector $\mathbf{x}$, there exists a QUT matrix $\mathbf{M}$ which simultaneously whitens the covariance and diagonalises an η-covariance matrix of $\mathbf{y} = \mathbf{M}^H\mathbf{x}$, i.e. $\mathbf{C}_\mathbf{y} = \mathbf{I}$ and $\mathbf{C}_{\mathbf{y}^\eta} = \mathbf{\Lambda}_\eta$.

Properness and quaternion strong uncorrelating transform

- **Properness:** an important statistical property in both $\mathbb{C}$ and $\mathbb{H}$.
- Characterised by the degree of correlation and/or power difference
- Properness depends on the pseudo-covariance in $\mathbb{C}$
- **Two types of properness in $\mathbb{H}$:**

  * $\mathbb{H}$-proper: if and only if all the η-covariance matrices vanish.

  * $\mathbb{C}^{\eta}$-proper for a single imaginary unit $\eta \in \{\imath, \jmath, \kappa\}$: if only the η-covariance matrix exists and the other two vanish.
    - $\mathbf{x}$ and $\mathbf{x}^{\eta}$ are correlated through the $E[\mathbf{x}(\mathbf{x}^{\eta})^{H}]$

    - The degree of $\mathbb{C}$-properness for a quaternion variable x is:

$$\rho = \frac{|E\{xx^{\eta H}\}|}{|E\{xx^{H}\}|} \quad \rho \in [0, 1]$$

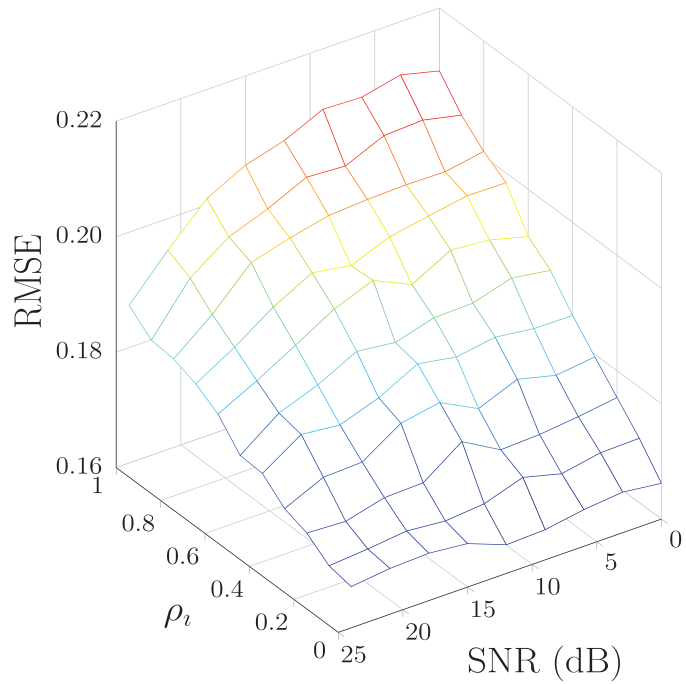
- For a $\mathbb{C}^{\eta}$-proper data, QUT can be considered as the quaternion strong uncorrelating transform (Q-SUT) which diagonalises the covariance and all three η-covariance matrices.

Simulation 1

Evaluate the performance accuracy in terms of

1. The additive white noise, SNR values

2. The $\imath$-circularity coefficient



Simulation 2: Alamouti-based coding

Assess the proposed QUT for a practical communication problem of Alamouti coding

- For a single-user, the model of two transmit antennas (a, b) and one receiver antenna:

$$\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{bmatrix} = \begin{bmatrix} \mathbf{s}_a & -\mathbf{s}_b^* \\ \mathbf{s}_b & \mathbf{s}_a^* \end{bmatrix} \begin{bmatrix} \Upsilon_a \\ \Upsilon_b \end{bmatrix} \in \mathbb{C} \tag{2}$$

Simulation 2: Alamouti-based coding

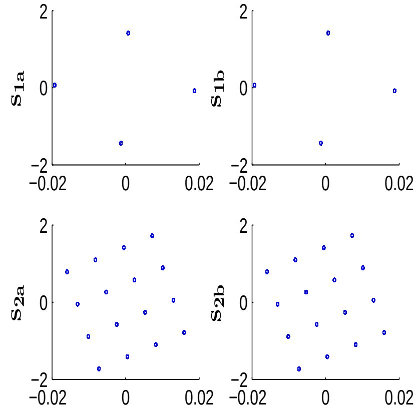
- Model for two users:

$$\begin{bmatrix}\mathbf{x}_{1a}\\ \mathbf{x}_{1b}\end{bmatrix} = \begin{bmatrix}\mathbf{s}_{1a} & -\mathbf{s}_{1b}^* \\ \mathbf{s}_{1b} & \mathbf{s}_{1a}^*\end{bmatrix}\begin{bmatrix}\Upsilon_{11a}\\ \Upsilon_{11b}\end{bmatrix} + \begin{bmatrix}\mathbf{s}_{2a} & -\mathbf{s}_{2b}^* \\ \mathbf{s}_{2b} & \mathbf{s}_{2a}^*\end{bmatrix}\begin{bmatrix}\Upsilon_{12a}\\ \Upsilon_{12b}\end{bmatrix} \in \mathbb{C}$$

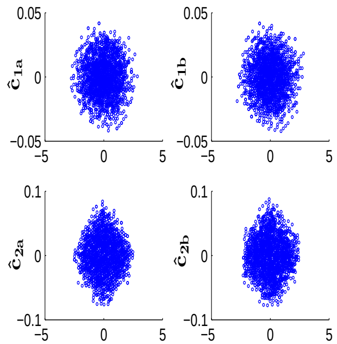
$$\begin{bmatrix}\mathbf{x}_{2a}\\ \mathbf{x}_{2b}\end{bmatrix} = \begin{bmatrix}\mathbf{s}_{1a} & -\mathbf{s}_{1b}^* \\ \mathbf{s}_{1b} & \mathbf{s}_{1a}^*\end{bmatrix}\begin{bmatrix}\Upsilon_{21a}\\ \Upsilon_{21b}\end{bmatrix} + \begin{bmatrix}\mathbf{s}_{2a} & -\mathbf{s}_{2b}^* \\ \mathbf{s}_{2b} & \mathbf{s}_{2a}^*\end{bmatrix}\begin{bmatrix}\Upsilon_{22a}\\ \Upsilon_{22b}\end{bmatrix} \in \mathbb{C} \quad (3)$$

$$\begin{bmatrix}\mathbf{x}_{1}\\ \mathbf{x}_{2}\end{bmatrix} = \begin{bmatrix}\Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22}\end{bmatrix}\begin{bmatrix}\mathbf{s}_{1}\\ \mathbf{s}_{2}\end{bmatrix} \in \mathbb{H} \qquad \begin{cases}\mathbf{x}_i = \mathbf{x}_{i_a} + \mathbf{x}_{i_b}\jmath \\ \mathbf{s}_i = \mathbf{s}_{i_a} + \mathbf{s}_{i_b}\jmath\end{cases}$$

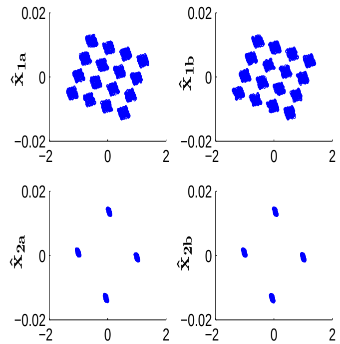
Quaternion-valued mixing matrix Υ and $\mathbf{s}_{1_a} = \mathbf{s}_{1_b}$



Original



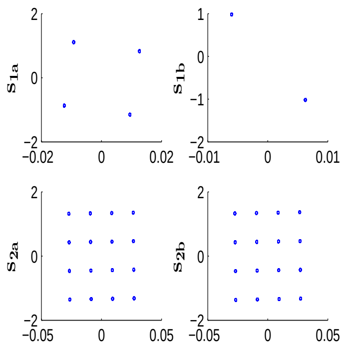
Complex



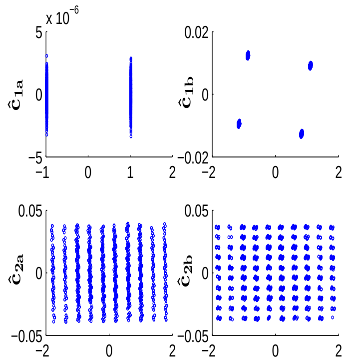
QUT

Complex-valued mixing matrix Υ and $\mathbf{s}_{1_a} \neq \mathbf{s}_{1_b}$

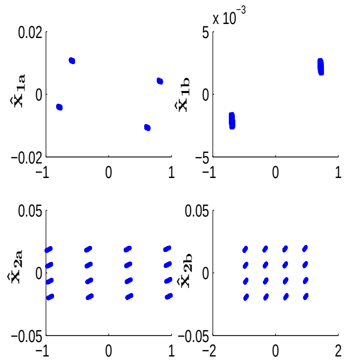
Original



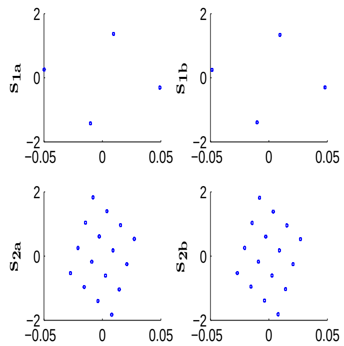
Complex



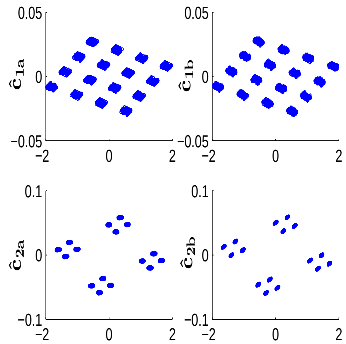
QUT



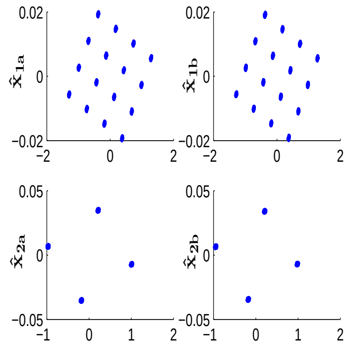
Complex-valued mixing matrix Υ and $s_{1_a} = s_{1_b}$



Original



Complex



QUT

Conclusion

1. A set of matrix decompositions for the joint diagonalisation of quaternion covariance matrices which satisfy:
    - $\mathbf{AB} = \mathbf{BA}$
    - normality of $\mathbf{AB}^{\eta}$
    - $\mathbf{BA} = \mathbf{A}^{\eta}\mathbf{B}$

2. MATLAB code available at my website[2]

3. Open problem 1: How to diagonalise a quaternion symmetric matrix $\mathbf{A} = \mathbf{A}^T$?

4. Open problem 2: How to simultaneously diagonalise three and four quaternion matrices with different structures?

[2] MATLAB code available at `http://www.surrey.ac.uk/cs/people/clive_cheong_took/index.htm`

References

Thank you for your kind attention...

[1] R. A. Horn and C. R. Johnson, "Matrix analysis." Cambridge university press, 2012.

[2] C. C. Took, D. P. Mandic, and F. Zhang, "On the unitary diagonalisation of a special class of quaternion matrices." Applied Mathematics Letters, vol. 24, no. 11, pp. 18061809, 2011.

[3] Horn, Roger A., and Fuzhen Zhang. "A generalization of the complex Autonne-Takagi factorization to quaternion matrices." Linear and Multilinear Algebra, vol. 60, no. 11-12, pp. 12391244, 2012.

[4] S. Sangwine and N. L. Bihan, "Quaternion singular value decomposition based on bidiagonalization to a real matrix using quaternion Householder transformations." Applied Mathematics and Computation, vol. 182, pp. 727738, 2006.

[5] J. Via, D. P. Palomar, L. Vielva, and I. Santamaria. "Quaternion ICA from second-order statistics." IEEE Transactions on Signal Processing, vol. 59, no. 4, pp. 1586-1600, 2011.

1. Generate three ARMA sources $\mathfrak{s}_d \in \mathbb{H}^{1\times N}, \quad d = \{1, 2, 3\}$
2. To obtain uncorrelated $\mathbb{C}^{\imath}$-proper sources, their real and imaginary parts were considered as real-valued signals and the following steps were applied:

1. $\mathfrak{S} = [\mathfrak{s}_{1_r}; \mathfrak{s}_{1_\imath}; \mathfrak{s}_{1_\jmath}; \mathfrak{s}_{1_\kappa}; \mathfrak{s}_{2_r}; \mathfrak{s}_{2_\imath}; \mathfrak{s}_{2_\jmath}; \mathfrak{s}_{2_\kappa}; \mathfrak{s}_{3_r}; \mathfrak{s}_{3_\imath}; \mathfrak{s}_{3_\jmath}; \mathfrak{s}_{3_\kappa}]$
2. $\mathbf{C}_{\mathfrak{S}} = \mathfrak{S}\mathfrak{S}^T$
3. $[\mathbf{U}, \mathbf{\Lambda}] = \text{svd}(\mathbf{C}_{\mathfrak{S}})$
4. $\tilde{\mathfrak{S}} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{U}^T\mathfrak{S}$
5. Produce the desired covariance matrix $\mathbf{C}_{\mathfrak{D}} \in \mathbb{R}^{12\times 12}$

$$\mathbf{C}_{\mathfrak{D}} = \left[\begin{array}{c|c|c} \mathbf{C}_{\mathfrak{s}} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_{\mathfrak{s}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathfrak{s}} \end{array}\right], \qquad \mathbf{C}_{\mathfrak{s}} = \begin{bmatrix} 1 & 0 & 0 & \rho_\imath \\ 0 & 1 & \rho_\imath & 0 \\ 0 & \rho_\imath & 1 & 0 \\ \rho_\imath & 0 & 0 & 1 \end{bmatrix}$$

   in which the zero matrix $\mathbf{0}$ is used to provide uncorrelatedness, and $\mathbf{C}_{\mathfrak{s}} \in \mathbb{R}^{4\times 4}$ is used to satisfy the $\mathbb{C}^{\imath}$-properness with specific $\imath$-circularity coefficient.
6. $[\mathbf{U}_{\mathfrak{D}}, \mathbf{\Lambda}_{\mathfrak{D}}] = \text{svd}(\mathbf{C}_{\mathfrak{D}})$
7. $\mathbf{S} = \mathbf{U}_{\mathfrak{D}}\mathbf{\Lambda}_{\mathfrak{D}}^{\frac{1}{2}}\tilde{\mathfrak{S}} = [\mathbf{s}_{1_r}; \mathbf{s}_{1_\imath}; \mathbf{s}_{1_\jmath}; \mathbf{s}_{1_\kappa}; \mathbf{s}_{2_r}; \mathbf{s}_{2_\imath}; \mathbf{s}_{2_\jmath}; \mathbf{s}_{2_\kappa}; \mathbf{s}_{3_r}; \mathbf{s}_{3_\imath}; \mathbf{s}_{3_\jmath}; \mathbf{s}_{3_\kappa}]$
8. $\mathbf{s}_d = \mathbf{s}_{d_r} + \imath\mathbf{s}_{d_\imath} + \jmath\mathbf{s}_{d_\jmath} + \kappa\mathbf{s}_{d_\kappa}, \quad d = \{1, 2, 3\}$

3. Using the above sources, generate inputs $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$

Consider three univariate quaternion-valued sources as:

$$\begin{cases} \mathbf{x} = \mathbf{x}_r + \imath\mathbf{x}_\imath + \jmath\mathbf{x}_\jmath + \kappa\mathbf{x}_\kappa \\ \mathbf{y} = \mathbf{y}_r + \imath\mathbf{y}_\imath + \jmath\mathbf{y}_\jmath + \kappa\mathbf{y}_\kappa \\ \mathbf{z} = \mathbf{z}_r + \imath\mathbf{z}_\imath + \jmath\mathbf{z}_\jmath + \kappa\mathbf{z}_\kappa \end{cases} \quad \in \mathbb{H}^{1\times N} \tag{4}$$

To obtain the real-valued covariance matrix, consider the quadrivariate correspondence of each source ($\mathbf{x}_\mathbb{R}$, $\mathbf{y}_\mathbb{R}$, $\mathbf{z}_\mathbb{R}$) and generate the 12×12 covariance matrix in $\mathbb{R}$:

$$\mathbf{W} = \begin{bmatrix} \mathbf{x}_\mathbb{R} \\ \mathbf{y}_\mathbb{R} \\ \mathbf{z}_\mathbb{R} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_\imath \\ \mathbf{x}_\jmath \\ \mathbf{x}_\kappa \\ \hline \mathbf{y}_r \\ \mathbf{y}_\imath \\ \mathbf{y}_\jmath \\ \mathbf{y}_\kappa \\ \hline \mathbf{z}_r \\ \mathbf{z}_\imath \\ \mathbf{z}_\jmath \\ \mathbf{z}_\kappa \end{bmatrix} \quad \rightarrow \quad \mathbf{C}_\mathbf{w} = E[\mathbf{W}\mathbf{W}^T] = E\left[\begin{array}{c|c|c} \mathbf{x}_\mathbb{R}\mathbf{x}_\mathbb{R}^T & \mathbf{x}_\mathbb{R}\mathbf{y}_\mathbb{R}^T & \mathbf{x}_\mathbb{R}\mathbf{z}_\mathbb{R}^T \\ \hline \mathbf{y}_\mathbb{R}\mathbf{x}_\mathbb{R}^T & \mathbf{y}_\mathbb{R}\mathbf{y}_\mathbb{R}^T & \mathbf{y}_\mathbb{R}\mathbf{z}_\mathbb{R}^T \\ \hline \mathbf{z}_\mathbb{R}\mathbf{x}_\mathbb{R}^T & \mathbf{z}_\mathbb{R}\mathbf{y}_\mathbb{R}^T & \mathbf{z}_\mathbb{R}\mathbf{z}_\mathbb{R}^T \end{array}\right] \tag{5}$$

$$
E\left[\begin{array}{cccc|cccc|cccc}
\mathbf{x}_r\mathbf{x}_r^T & \mathbf{x}_r\mathbf{x}_\imath^T & \mathbf{x}_r\mathbf{x}_\jmath^T & \mathbf{x}_r\mathbf{x}_\kappa^T & \mathbf{x}_r\mathbf{y}_r^T & \mathbf{x}_r\mathbf{y}_\imath^T & \mathbf{x}_r\mathbf{y}_\jmath^T & \mathbf{x}_r\mathbf{y}_\kappa^T & \mathbf{x}_r\mathbf{z}_r^T & \mathbf{x}_r\mathbf{z}_\imath^T & \mathbf{x}_r\mathbf{z}_\jmath^T & \mathbf{x}_r\mathbf{z}_\kappa^T \\
\mathbf{x}_\imath\mathbf{x}_r^T & \mathbf{x}_\imath\mathbf{x}_\imath^T & \mathbf{x}_\imath\mathbf{x}_\jmath^T & \mathbf{x}_\imath\mathbf{x}_\kappa^T & \mathbf{x}_\imath\mathbf{y}_r^T & \mathbf{x}_\imath\mathbf{y}_\imath^T & \mathbf{x}_\imath\mathbf{y}_\jmath^T & \mathbf{x}_\imath\mathbf{y}_\kappa^T & \mathbf{x}_\imath\mathbf{z}_r^T & \mathbf{x}_\imath\mathbf{z}_\imath^T & \mathbf{x}_\imath\mathbf{z}_\jmath^T & \mathbf{x}_\imath\mathbf{z}_\kappa^T \\
\mathbf{x}_\jmath\mathbf{x}_r^T & \mathbf{x}_\jmath\mathbf{x}_\imath^T & \mathbf{x}_\jmath\mathbf{x}_\jmath^T & \mathbf{x}_\jmath\mathbf{x}_\kappa^T & \mathbf{x}_\jmath\mathbf{y}_r^T & \mathbf{x}_\jmath\mathbf{y}_\imath^T & \mathbf{x}_\jmath\mathbf{y}_\jmath^T & \mathbf{x}_\jmath\mathbf{y}_\kappa^T & \mathbf{x}_\jmath\mathbf{z}_r^T & \mathbf{x}_\jmath\mathbf{z}_\imath^T & \mathbf{x}_\jmath\mathbf{z}_\jmath^T & \mathbf{x}_\jmath\mathbf{z}_\kappa^T \\
\mathbf{x}_\kappa\mathbf{x}_r^T & \mathbf{x}_\kappa\mathbf{x}_\imath^T & \mathbf{x}_\kappa\mathbf{x}_\jmath^T & \mathbf{x}_\kappa\mathbf{x}_\kappa^T & \mathbf{x}_\kappa\mathbf{y}_r^T & \mathbf{x}_\kappa\mathbf{y}_\imath^T & \mathbf{x}_\kappa\mathbf{y}_\jmath^T & \mathbf{x}_\kappa\mathbf{y}_\kappa^T & \mathbf{x}_\kappa\mathbf{z}_r^T & \mathbf{x}_\kappa\mathbf{z}_\imath^T & \mathbf{x}_\kappa\mathbf{z}_\jmath^T & \mathbf{x}_\kappa\mathbf{z}_\kappa^T \\
\hline
\mathbf{y}_r\mathbf{x}_r^T & \mathbf{y}_r\mathbf{x}_\imath^T & \mathbf{y}_r\mathbf{x}_\jmath^T & \mathbf{y}_r\mathbf{x}_\kappa^T & \mathbf{y}_r\mathbf{y}_r^T & \mathbf{y}_r\mathbf{y}_\imath^T & \mathbf{y}_r\mathbf{y}_\jmath^T & \mathbf{y}_r\mathbf{y}_\kappa^T & \mathbf{y}_r\mathbf{z}_r^T & \mathbf{y}_r\mathbf{z}_\imath^T & \mathbf{y}_r\mathbf{z}_\jmath^T & \mathbf{y}_r\mathbf{z}_\kappa^T \\
\mathbf{y}_\imath\mathbf{x}_r^T & \mathbf{y}_\imath\mathbf{x}_\imath^T & \mathbf{y}_\imath\mathbf{x}_\jmath^T & \mathbf{y}_\imath\mathbf{x}_\kappa^T & \mathbf{y}_\imath\mathbf{y}_r^T & \mathbf{y}_\imath\mathbf{y}_\imath^T & \mathbf{y}_\imath\mathbf{y}_\jmath^T & \mathbf{y}_\imath\mathbf{y}_\kappa^T & \mathbf{y}_\imath\mathbf{z}_r^T & \mathbf{y}_\imath\mathbf{z}_\imath^T & \mathbf{y}_\imath\mathbf{z}_\jmath^T & \mathbf{y}_\imath\mathbf{z}_\kappa^T \\
\mathbf{y}_\jmath\mathbf{x}_r^T & \mathbf{y}_\jmath\mathbf{x}_\imath^T & \mathbf{y}_\jmath\mathbf{x}_\jmath^T & \mathbf{y}_\jmath\mathbf{x}_\kappa^T & \mathbf{y}_\jmath\mathbf{y}_r^T & \mathbf{y}_\jmath\mathbf{y}_\imath^T & \mathbf{y}_\jmath\mathbf{y}_\jmath^T & \mathbf{y}_\jmath\mathbf{y}_\kappa^T & \mathbf{y}_\jmath\mathbf{z}_r^T & \mathbf{y}_\jmath\mathbf{z}_\imath^T & \mathbf{y}_\jmath\mathbf{z}_\jmath^T & \mathbf{y}_\jmath\mathbf{z}_\kappa^T \\
\mathbf{y}_\kappa\mathbf{x}_r^T & \mathbf{y}_\kappa\mathbf{x}_\imath^T & \mathbf{y}_\kappa\mathbf{x}_\jmath^T & \mathbf{y}_\kappa\mathbf{x}_\kappa^T & \mathbf{y}_\kappa\mathbf{y}_r^T & \mathbf{y}_\kappa\mathbf{y}_\imath^T & \mathbf{y}_\kappa\mathbf{y}_\jmath^T & \mathbf{y}_\kappa\mathbf{y}_\kappa^T & \mathbf{y}_\kappa\mathbf{z}_r^T & \mathbf{y}_\kappa\mathbf{z}_\imath^T & \mathbf{y}_\kappa\mathbf{z}_\jmath^T & \mathbf{y}_\kappa\mathbf{z}_\kappa^T \\
\hline
\mathbf{z}_r\mathbf{x}_r^T & \mathbf{z}_r\mathbf{x}_\imath^T & \mathbf{z}_r\mathbf{x}_\jmath^T & \mathbf{z}_r\mathbf{x}_\kappa^T & \mathbf{z}_r\mathbf{y}_r^T & \mathbf{z}_r\mathbf{y}_\imath^T & \mathbf{z}_r\mathbf{y}_\jmath^T & \mathbf{z}_r\mathbf{y}_\kappa^T & \mathbf{z}_r\mathbf{z}_r^T & \mathbf{z}_r\mathbf{z}_\imath^T & \mathbf{z}_r\mathbf{z}_\jmath^T & \mathbf{z}_r\mathbf{z}_\kappa^T \\
\mathbf{z}_\imath\mathbf{x}_r^T & \mathbf{z}_\imath\mathbf{x}_\imath^T & \mathbf{z}_\imath\mathbf{x}_\jmath^T & \mathbf{z}_\imath\mathbf{x}_\kappa^T & \mathbf{z}_\imath\mathbf{y}_r^T & \mathbf{z}_\imath\mathbf{y}_\imath^T & \mathbf{z}_\imath\mathbf{y}_\jmath^T & \mathbf{z}_\imath\mathbf{y}_\kappa^T & \mathbf{z}_\imath\mathbf{z}_r^T & \mathbf{z}_\imath\mathbf{z}_\imath^T & \mathbf{z}_\imath\mathbf{z}_\jmath^T & \mathbf{z}_\imath\mathbf{z}_\kappa^T \\
\mathbf{z}_\jmath\mathbf{x}_r^T & \mathbf{z}_\jmath\mathbf{x}_\imath^T & \mathbf{z}_\jmath\mathbf{x}_\jmath^T & \mathbf{z}_\jmath\mathbf{x}_\kappa^T & \mathbf{z}_\jmath\mathbf{y}_r^T & \mathbf{z}_\jmath\mathbf{y}_\imath^T & \mathbf{z}_\jmath\mathbf{y}_\jmath^T & \mathbf{z}_\jmath\mathbf{y}_\kappa^T & \mathbf{z}_\jmath\mathbf{z}_r^T & \mathbf{z}_\jmath\mathbf{z}_\imath^T & \mathbf{z}_\jmath\mathbf{z}_\jmath^T & \mathbf{z}_\jmath\mathbf{z}_\kappa^T \\
\mathbf{z}_\kappa\mathbf{x}_r^T & \mathbf{z}_\kappa\mathbf{x}_\imath^T & \mathbf{z}_\kappa\mathbf{x}_\jmath^T & \mathbf{z}_\kappa\mathbf{x}_\kappa^T & \mathbf{z}_\kappa\mathbf{y}_r^T & \mathbf{z}_\kappa\mathbf{y}_\imath^T & \mathbf{z}_\kappa\mathbf{y}_\jmath^T & \mathbf{z}_\kappa\mathbf{y}_\kappa^T & \mathbf{z}_\kappa\mathbf{z}_r^T & \mathbf{z}_\kappa\mathbf{z}_\imath^T & \mathbf{z}_\kappa\mathbf{z}_\jmath^T & \mathbf{z}_\kappa\mathbf{z}_\kappa^T
\end{array}\right] \tag{6}
$$

If three uncorrelated $\mathbb{C}^{\imath}$-proper sources are generated, the following observations on the structure of the covariance can be made:

- The off diagonal blocks in $\mathbf{C_w}$ are zero, since sources are uncorrelated with regard to each other.
- The diagonal blocks have the following algebraic structure in order to satisfy the $\mathbb{C}^{\imath}$-properness condition with specific values ρ.

$$\mathbf{C}_{\mathfrak{s}} = \begin{bmatrix} 1 & 0 & 0 & \rho \\ 0 & 1 & \rho & 0 \\ 0 & \rho & 1 & 0 \\ \rho & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Based on the matrix $\mathbf{C}_{\mathfrak{s}}$, the covariance matrices of each source are defined as:

$$\mathbf{C_q} = 4\mathbf{I}, \quad \mathbf{C}_{\mathbf{q}^{\imath}} = \kappa(4\rho)\mathbf{I}, \quad \mathbf{C}_{\mathbf{q}^{\jmath}} = \mathbf{C}_{\mathbf{q}^{\kappa}} = \mathbf{0}, \quad \mathbf{q} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \tag{8}$$

The final structure of the covariance matrix is given by:

$$\mathbf{C_w} = \left[\begin{array}{c|c|c} \mathbf{C}_{\mathfrak{s}} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_{\mathfrak{s}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathfrak{s}} \end{array}\right] \tag{9}$$

Table 1: The standard and η-covariance matrices of $\mathbf{q}$ in terms of its real components.

	$\mathbf{C}_{\mathbf{q}}$	$\mathbf{C}_{\mathbf{q}^{\imath}}$
$\Re[\cdot]$	$E[\mathbf{q}_r\mathbf{q}_r^T] + E[\mathbf{q}_\imath\mathbf{q}_\imath^T] + E[\mathbf{q}_\jmath\mathbf{q}_\jmath^T] + E[\mathbf{q}_\kappa\mathbf{q}_\kappa^T]$	$E[\mathbf{q}_r\mathbf{q}_r^T] + E[\mathbf{q}_\imath\mathbf{q}_\imath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\jmath^T] - E[\mathbf{q}_\kappa\mathbf{q}_\kappa^T]$
$\Im_\imath[\cdot]$	$E[\mathbf{q}_\imath\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\imath^T] + E[\mathbf{q}_\kappa\mathbf{q}_\jmath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\kappa^T]$	$E[\mathbf{q}_\imath\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\imath^T] + E[\mathbf{q}_\jmath\mathbf{q}_\kappa^T] - E[\mathbf{q}_\kappa\mathbf{q}_\jmath^T]$
$\Im_\jmath[\cdot]$	$E[\mathbf{q}_\jmath\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\jmath^T] + E[\mathbf{q}_\imath\mathbf{q}_\kappa^T] - E[\mathbf{q}_\kappa\mathbf{q}_\imath^T]$	$E[\mathbf{q}_r\mathbf{q}_\jmath^T] + E[\mathbf{q}_\jmath\mathbf{q}_r^T] - E[\mathbf{q}_\kappa\mathbf{q}_\imath^T] - E[\mathbf{q}_\imath\mathbf{q}_\kappa^T]$
$\Im_\kappa[\cdot]$	$E[\mathbf{q}_\kappa\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\kappa^T] + E[\mathbf{q}_\jmath\mathbf{q}_\imath^T] - E[\mathbf{q}_\imath\mathbf{q}_\jmath^T]$	$E[\mathbf{q}_\kappa\mathbf{q}_r^T] + E[\mathbf{q}_r\mathbf{q}_\kappa^T] + E[\mathbf{q}_\imath\mathbf{q}_\jmath^T] + E[\mathbf{q}_\jmath\mathbf{q}_\imath^T]$
	$\mathbf{C}_{\mathbf{q}^{\jmath}}$	$\mathbf{C}_{\mathbf{q}^{\kappa}}$
$\Re[\cdot]$	$E[\mathbf{q}_r\mathbf{q}_r^T] - E[\mathbf{q}_\imath\mathbf{q}_\imath^T] + E[\mathbf{q}_\jmath\mathbf{q}_\jmath^T] - E[\mathbf{q}_\kappa\mathbf{q}_\kappa^T]$	$E[\mathbf{q}_r\mathbf{q}_r^T] - E[\mathbf{q}_\imath\mathbf{q}_\imath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\jmath^T] + E[\mathbf{q}_\kappa\mathbf{q}_\kappa^T]$
$\Im_\imath[\cdot]$	$E[\mathbf{q}_\imath\mathbf{q}_r^T] + E[\mathbf{q}_r\mathbf{q}_\imath^T] + E[\mathbf{q}_\kappa\mathbf{q}_\jmath^T] + E[\mathbf{q}_\jmath\mathbf{q}_\kappa^T]$	$E[\mathbf{q}_\imath\mathbf{q}_r^T] + E[\mathbf{q}_r\mathbf{q}_\imath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\kappa^T] - E[\mathbf{q}_\kappa\mathbf{q}_\jmath^T]$
$\Im_\jmath[\cdot]$	$E[\mathbf{q}_\jmath\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\jmath^T] - E[\mathbf{q}_\kappa\mathbf{q}_\imath^T] + E[\mathbf{q}_\imath\mathbf{q}_\kappa^T]$	$E[\mathbf{q}_r\mathbf{q}_\jmath^T] + E[\mathbf{q}_\jmath\mathbf{q}_r^T] + E[\mathbf{q}_\kappa\mathbf{q}_\imath^T] + E[\mathbf{q}_\imath\mathbf{q}_\kappa^T]$
$\Im_\kappa[\cdot]$	$E[\mathbf{q}_\kappa\mathbf{q}_r^T] + E[\mathbf{q}_r\mathbf{q}_\kappa^T] - E[\mathbf{q}_\imath\mathbf{q}_\jmath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\imath^T]$	$E[\mathbf{q}_\kappa\mathbf{q}_r^T] - E[\mathbf{q}_r\mathbf{q}_\kappa^T] + E[\mathbf{q}_\imath\mathbf{q}_\jmath^T] - E[\mathbf{q}_\jmath\mathbf{q}_\imath^T]$