BIPARTITE SUBGRAPH DECOMPOSITION FOR CRITICALLY SAMPLED WAVELET FILTERBANKS ON ARBITARY GRAPHS

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Introduction

Why graph wavelets?

- Graph: describe data structures in various scenarios
- Key problem: graph wavelets for compact representation

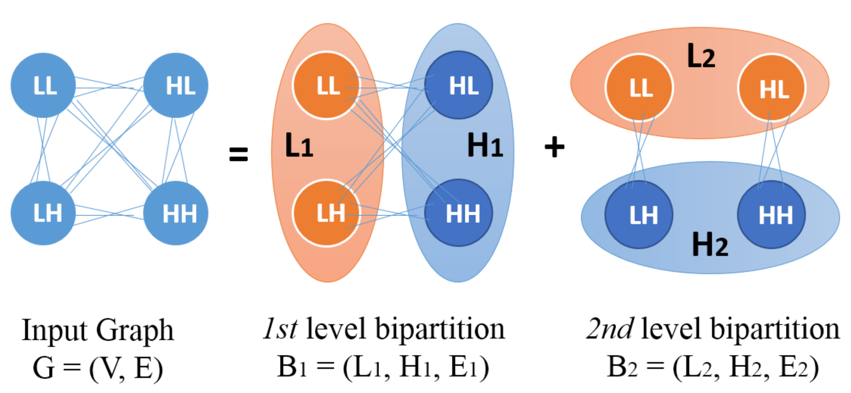
Why bipartite subgraph decomposition?

- Limit of recent works: GraphBior [1], only for bipartite graph
- Solution for non-bipartite graph: decompose into bipartite subgraphs, see *Fig. 1*

Problem Statement

- Bipartite subgraph decomposition for compact signal representation
- Previous methods [2][3][4] neglect relation between metrics and energy compactness
- Proposed method: a) minimize the mid-frequency multiplicity; b) maximize the structure preservation

Fig. 1 Example of Bipartite Subgraph Decomposition



Graph Wavelet Filterbanks

Graph Spectrum and Spectral Filter

- Laplacian matrix L = D - W; D is degree matrix, W is adjacency matrix
- Normalized form $\mathcal{L} = D^{-1/2} L D^{-1/2}$: eigenvalues $\{\lambda_i\}$ within range [0, 2], interpreted as graph spectrum
- Spectral Filter: defined with spectral kernel h(λ)

Critically Sampled Wavelet Filterbanks — for Bipartite Graph-Signal

- Flowchart *Fig. 2*: decompose *f* into low-pass and high-pass components
- **H** and **G**: based on frequency folding, with spectral kernels in *Fig. 3*
- λ = 1: minimal energy discrimination
- Bipartite subgraph decomposition required for non-bipartite graph-signals

Fig.2 Two-channel wavelet filterbanks on bipartite graph B = (L, H, E)

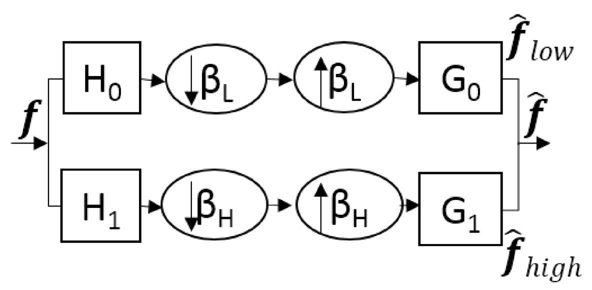
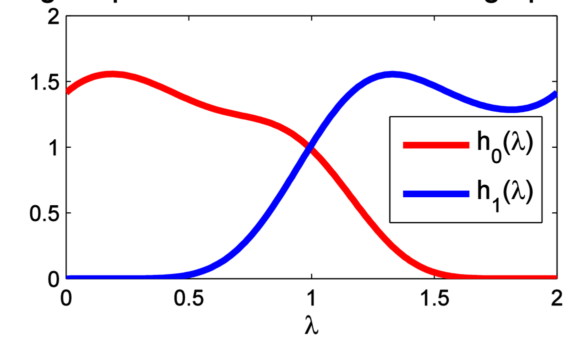


Fig.3 Spectral kernels of H0 H1 in graphBior

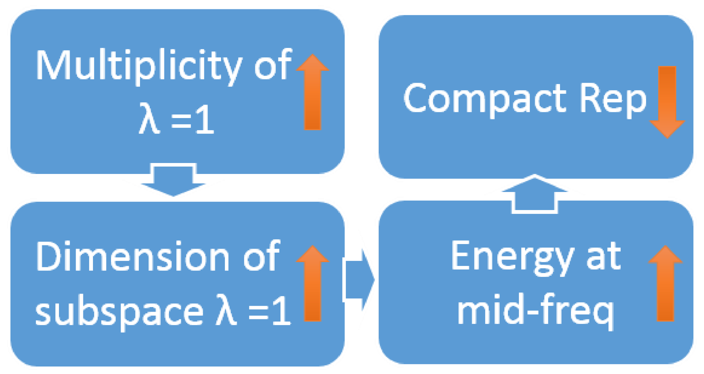


Proposed Method

- **Goal**: compact representation of signals in original graph G projected to wavelet domain of bipartite subgraph G'
- **Criteria**: minimum mid-frequency multiplicity & maximum structure preservation

Minimum Mid-Frequency Multiplicity

Fig.4 How mid-frequency multiplicity affects compact representation



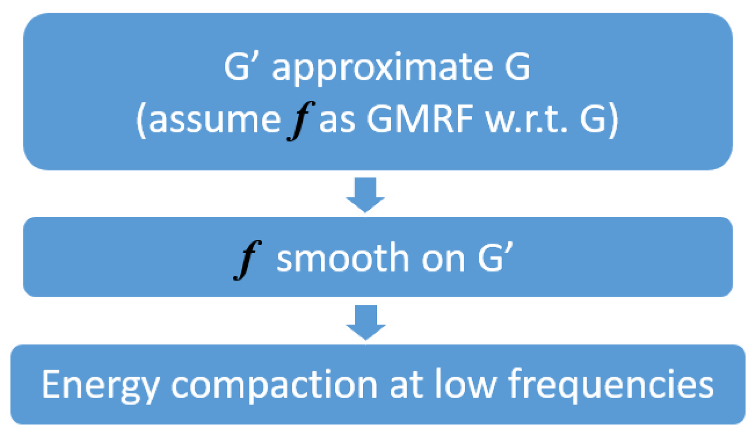
- Table 1 exhibits high multiplicity of λ = 1 for first level bipartite subgraph using Harary's [2] in real-world cases.

Graph	Vertex number	Multiplicity
Minnesota traffic graph	2642	428
Yale Coat of Arms	1059	103
China Temperature Graph	208	32

- Multiplicity of λ = 1 is equivalent to null(W)
- **Measurement: rank(W)**

Maximum Structure Preservation

Fig. 5 How structure preservation leads to compact representation



How to measure structure preservation?

- **KL Divergence**: measure graph difference
- GMRF w.r.t. G ~ $\mathcal{N}(\mu, \Sigma)$, $\Sigma^{-1} = L + \delta I$;

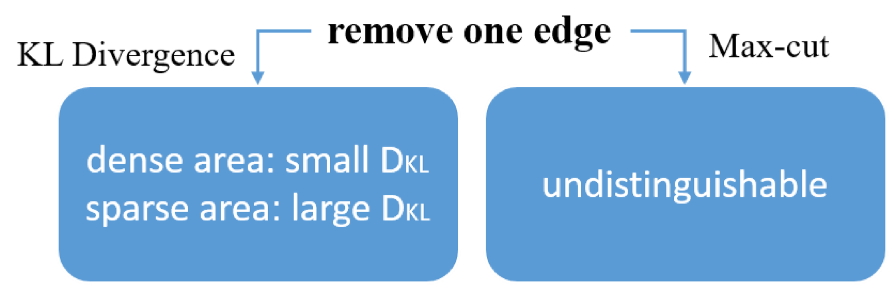
  G' ~ $\mathcal{N}_R(\mu_R, \Sigma_R)$, $\Sigma_R^{-1} = L_R + \delta I$:

$$D_{KL}(\mathcal{N}||\mathcal{N}_R) = \frac{1}{2}\left(\text{tr}(\Sigma_R^{-1}\Sigma) + (\mu_R - \mu)^T \Sigma_R^{-1}(\mu_R - \mu) - N + \ln\left(\frac{|\Sigma_R|}{|\Sigma|}\right)\right)$$

Advantage over Max-cut

- [3][4] use max-cut as measurement
- *Fig.6* shows max-cut's dilemma can be solved by KL Divergence.

Fig. 6 KL Divergence vs Max-cut



References: [1] S.N. and A.O. "Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs," TSP'13.

[2] S.N. and A.O. "Perfect reconstruction two-channel wavelet filter banks for graph structured data," TSP'12.

[3] S.N. and A.O. "Multi-dimensional separable critically sampled wavelet filterbanks on arbitrary graphs," ICASSP'12.

[4] H.N. and M.Do "Downsampling of signals on graphs via maximum spanning trees," TSP'15.

Algorithm (MFS)

- maximizing rank(W) ≠ minimizing D_{KL}
- **Proposed algorithm**: Bipartite Subgraph Decomposition Optimizing Mid-frequency and Structure (MFS), summed up as follows:

Input: graph $\mathcal{G}$, decomposition level k
Output: edge-disjoint bipartite graphs $\mathcal{B}_1,...,\mathcal{B}_k$

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1: for i = 1:k do
2:    Find connected components in G.
3:    For each component, put the starting vertex in set 1.
4:    Use breadth-first search to explore other vertices, and choose
   the proper set by jointly comparing rank(W_{1,2}) and D_KL.
5:    After all vertices are discovered, bipartite graph B_i is given.
6:    Update G by removing edges in B_i.
7: end for
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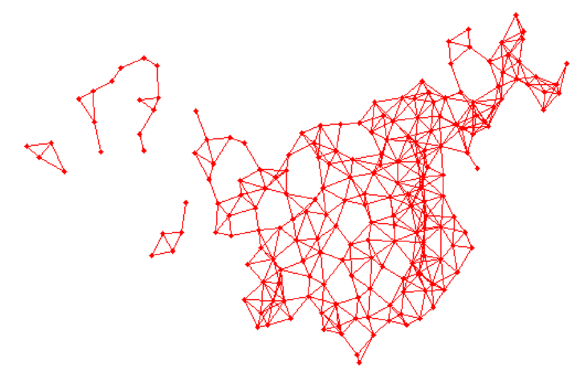
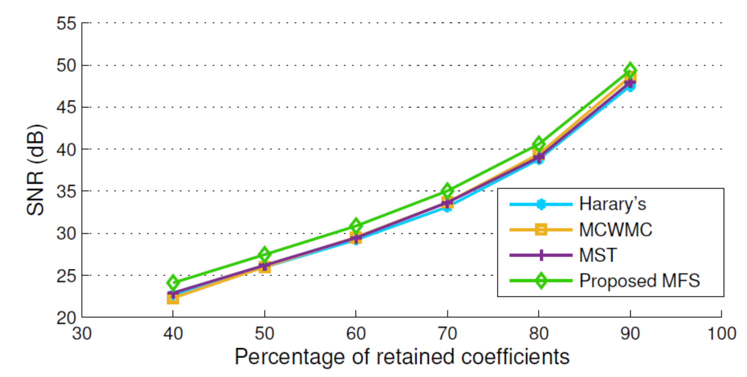
Experiments

- Steps: 1) bipartite subgraph decomposition; 2) GraphBior[1]; 3) reconstruct the signal with n% largest wavelet coefficients
- China temperature graph: monthly average temperature from Oct.09 to May12, vertices connected to neighbors with distance < threshold T





- Table 2: Average gain of proposed MFS over competing schemes in SNR(dB) for graphs with different connections: column 2~5, threshold from T to 1.4 T; column 6~8, vertices connected to knn with k = 7, 8, 9. It shows MFS outperforms existing schemes in all different graphs.

	T	0.8T	1.2T	1.4T	k=7	k=8	k=9
Harary's[2]	1.65	1.43	0.82	0.82	0.76	0.64	1.34
MCWMC[3]	1.35	0.74	1.17	1.24	1.56	1.62	2.06
MST[4]	1.35	0.16	2.24	1.38	0.93	0.64	1.91

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