Recursive versions of the reassigned spectrogram and of the synchrosqueezed STFT

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Introduction

Goals of this study

Compute time-frequency representations

- candidate for real-time implementation
- allow adjustments from the user through the damping parameter μ (Levenberg-Marquardt approach)
- allow modes separation and signal reconstruction (synchrosqueezing)
- filter bank approach

Proposed methodology

- special case of the STFT using a causal, infinite length window function that can be rewritten as a causal IIR recursive filtering
- the algorithmic complexity depends on the filter order and on the analyzed frequency bandwidth

Filter-based reassigned and synchrosqueezed STFT

The STFT as a convolution product

The STFT of a signal x using a real-valued analysis window h, denoted $F_x^h(t,\omega) = M_x^h(t,\omega)\,\mathrm{e}^{j\Phi_x^h(t,\omega)}$ can be related to the linear convolution product between the analyzed signal x and the complex valued impulse response of a bandpass filter $g(t,\omega) = h(t)\,\mathrm{e}^{j\omega t}$:

$$y_x^g(t,\omega) = \int_{-\infty}^{+\infty} g(\tau,\omega)x(t-\tau)\,d\tau = |y_x^g(t,\omega)|\,\mathrm{e}^{j\Psi_x^g(t,\omega)}$$
$$= F_x^h(t,\omega)\,\mathrm{e}^{j\omega t} = M_x^h(t,\omega)\,\mathrm{e}^{j(\Phi_x^h(t,\omega)+\omega t)} \quad (1)$$

Thus, $M_x^h(t,\omega) = |y_x^g(t,\omega)|$ and $\Phi_x^h(t,\omega) = \Psi_x^g(t,\omega) - \omega t$.

Rewording the reassignment operators of the spectrogram

According to [1], the spectrogram reassignment operators can be reformulated using the phase of $y_x^g(t,\omega)$, denoted $\Psi_x^g(t,\omega) = \Phi_x^h(t,\omega) + \omega t$

$$\hat{t}(t,\omega) = -\frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) = t - \frac{\partial \Psi_x^g}{\partial \omega}(t,\omega), \quad (2)$$

$$\hat{\omega}(t,\omega) = \omega + \frac{\partial \Phi_x^h}{\partial t}(t,\omega) = \frac{\partial \Psi_x^g}{\partial t}(t,\omega). \quad (3)$$

The reassigned spectrogram is expressed as

$$\mathrm{RSP}(t,\omega) = \iint_{\mathbb{R}^2} |y_x^g(t',\omega')|^2 \delta(t-\hat{t}(t',\omega'))\,\delta(\omega-\hat{\omega}(t',\omega'))\,dt'd\omega' \quad (4)$$

where $\delta(t)$ denotes the Dirac distribution.

Rewording the Levenberg-Marquardt reassignment [2]

$$\begin{pmatrix}\tilde{t}(t,\omega)\\ \tilde{\omega}(t,\omega)\end{pmatrix} = \begin{pmatrix} t\\ \omega\end{pmatrix} - \left(\nabla^t R_x^h(t,\omega) + \mu I_2\right)^{-1} R_x^h(t,\omega) \quad (5)$$

with $R_x^h(t,\omega) = \begin{pmatrix} t-\hat{t}(t,\omega)\\ \omega-\hat{\omega}(t,\omega)\end{pmatrix} = \begin{pmatrix} \frac{\partial \Psi_x^g}{\partial \omega}(t,\omega)\\ \omega - \frac{\partial \Psi_x^g}{\partial t}(t,\omega)\end{pmatrix}$

$$\nabla^t R_x^h(t,\omega) = \begin{pmatrix} \frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega) & \frac{\partial^2 \Psi_x^g}{\partial \omega^2}(t,\omega)\\ -\frac{\partial^2 \Psi_x^g}{\partial t^2}(t,\omega) & 1 - \frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega)\end{pmatrix}$$

Hence, the Levenberg-Marquardt reassigned spectrogram (LMRSP(t,ω)) is obtained by replacing $(\hat{t},\hat{\omega})$ by $(\tilde{t},\tilde{\omega})$ in Eq. (4).

Rewording the partial derivatives of the phase

$$\frac{\partial \Psi_x^g}{\partial t}(t,\omega) = \mathrm{Im}\left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right)$$

$$\frac{\partial \Psi_x^g}{\partial \omega}(t,\omega) = \mathrm{Re}\left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right)$$

$$\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega) = \mathrm{Re}\left(\frac{y_x^{\mathcal{DT}g}(t,\omega)}{y_x^g(t,\omega)} - \frac{y_x^{\mathcal{D}g}(t,\omega)y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)^2}\right)$$

$$\frac{\partial^2 \Psi_x^g}{\partial t^2}(t,\omega) = \mathrm{Im}\left(\frac{y_x^{\mathcal{D}^2g}(t,\omega)}{y_x^g(t,\omega)} - \left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right)$$

$$\frac{\partial^2 \Psi_x^g}{\partial \omega^2}(t,\omega) = -\mathrm{Im}\left(\frac{y_x^{\mathcal{T}^2g}(t,\omega)}{y_x^g(t,\omega)} - \left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right)$$

where y_x^g, $y_x^{\mathcal{T}g}$, $y_x^{\mathcal{D}g}$, $y_x^{\mathcal{DT}g}$, $y_x^{\mathcal{T}^2g}$ and $y_x^{\mathcal{D}^2g}$ are the outputs of the filters using respectively the impulse responses $g(t,\omega)$, $\mathcal{T}g = t\,g(t,\omega)$, $\mathcal{D}g(t,\omega) = \frac{\partial g}{\partial t}(t,\omega)$, $\mathcal{DT}g(t,\omega) = \frac{\partial}{\partial t}(t\,g(t,\omega))$, $\mathcal{T}^2g(t,\omega) = t^2 g(t,\omega)$ and $\mathcal{D}^2g(t,\omega) = \frac{\partial^2 g}{\partial t^2}(t,\omega)$.

Rewording the synchrosqueezed STFT

As previously defined, y_x^g admits the following signal reconstruction formula

$$x(t-t_0) = \frac{1}{h(t_0)}\int_{-\infty}^{+\infty} y_x^g(t,\omega)\,\mathrm{e}^{-j\omega t_0}\frac{d\omega}{2\pi}, \text{ when } h(t_0) \neq 0. \quad (6)$$

Synchrosqueezed STFT [5]

$$\mathrm{Sy}_x^g(t,\omega) = \int_{\mathbb{R}} y_x^g(t,\omega')\,\mathrm{e}^{-j\omega' t_0}\delta(\omega - \hat{\omega}(t,\omega'))\,d\omega' \quad (7)$$

LMSy$_x^g(t,\omega)$ is obtained by replacing $\hat{\omega}$ by $\tilde{\omega}$. A sharpen time-frequency representation is provided by $|\mathrm{Sy}_x^g(t,\omega)|^2$. The signal can be reconstructed from $\mathrm{Sy}_x^g(t,\omega)$ as

$$\hat{x}(t-t_0) = \frac{1}{h(t_0)}\int_{-\infty}^{+\infty} \mathrm{Sy}_x^g(t,\omega)\frac{d\omega}{2\pi} \quad (8)$$

Towards a recursive implementation

Recursive implementation

According to [3], y_x^g can be recursively implemented using

$$h_k(t) = \frac{t^{k-1}}{T^k(k-1)!}\,\mathrm{e}^{-t/T}\,U(t), \quad (9)$$

$$g_k(t,\omega) = h_k(t)\,\mathrm{e}^{j\omega t} = \frac{t^{k-1}}{T^k(k-1)!}\,\mathrm{e}^{pt}\,U(t) \quad (10)$$

with $p = -\frac{1}{T} + j\omega$, $k \geq 1$ being the filter order, T the time spread of the window and $U(t)$ the Heaviside step function.

Discretization using the impulse invariance method [4]

$$G_k(z,\omega) = T_s\mathcal{Z}\left\{g_k(t,\omega)\right\} = \frac{\sum_{i=0}^{k-1} b_i z^{-i}}{1 + \sum_{i=1}^{k} a_i z^{-i}} \quad (11)$$

with $b_i = \frac{1}{L^k(k-1)!}B_{k-1,k-i-1}\alpha^i$, $\alpha = \mathrm{e}^{pT_s}$, $L = T/T_s$, $a_i = A_{k,i}\left(-\alpha\right)^i$, T_s being the sampling period. $B_{k,i} = \sum_{j=0}^{i}(-1)^j A_{k+1,j}(i+1-j)^k$ denotes the Eulerian numbers and $A_{k,i}$ the binomial coefficients.
Hence, using $y_k[n,m] \approx y_x^{g_k}(nT_s, \frac{2\pi m}{MT_s})$ with $n \in \mathbb{Z}$ and $m = 0, 1, \ldots, M-1$, we obtain

$$y_k[n,m] = \sum_{i=0}^{k-1} b_i\,x[n-i] - \sum_{i=1}^{k} a_i\,y_k[n-i,m] \quad (12)$$

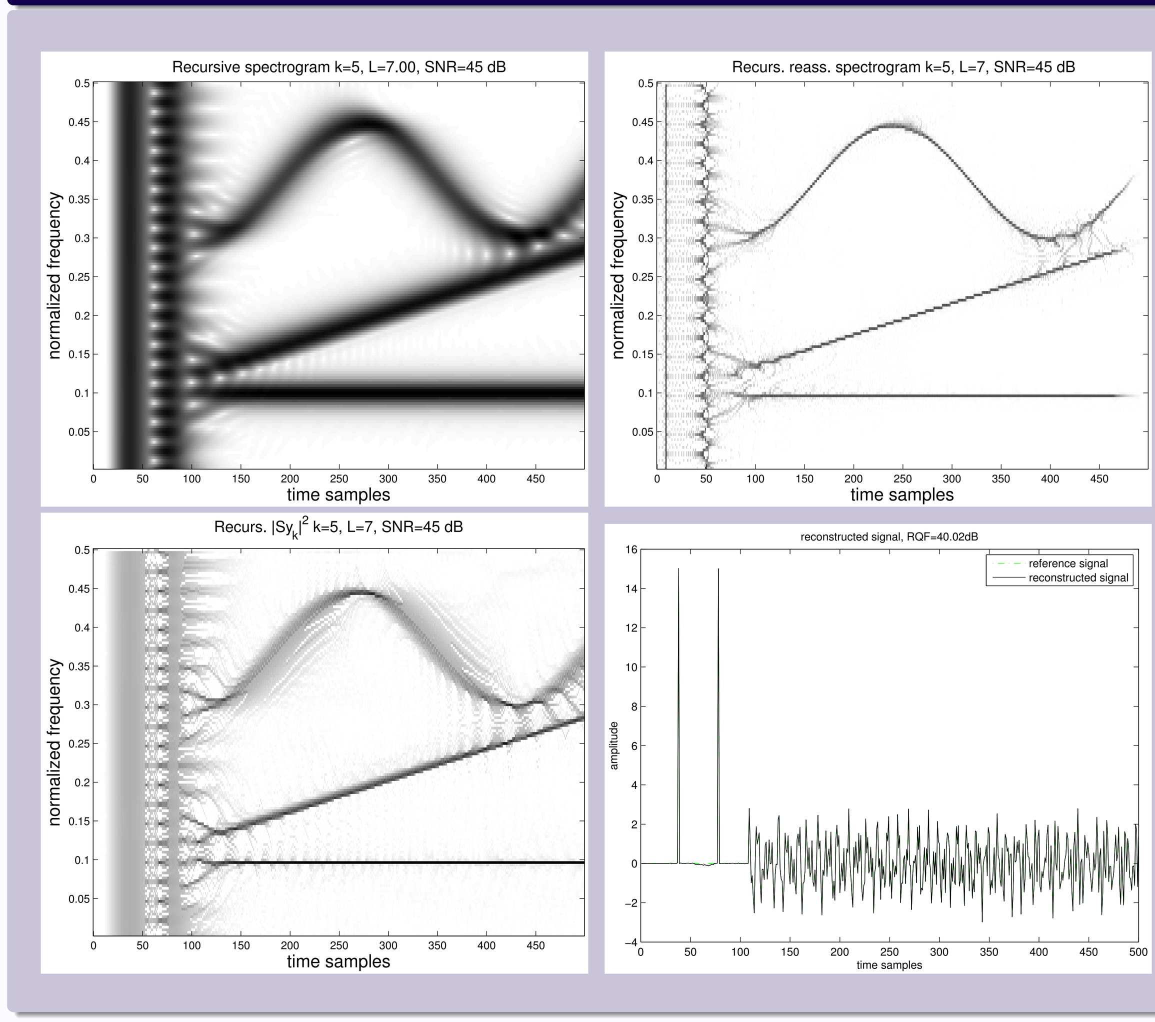
Algorithm implementation for recursive TFR computation

At time index n :

1. Compute the required $y_k^g[n,m]$ using $x[n-i]$ and $y_k[n-j,m]$ with $i \in [0, k-1]$, $j \in [1,k]$
2. Compute the other required specific filtered signals (*i.e.* $y_k^{\mathcal{T}g}$, $y_k^{\mathcal{D}g}$, $y_k^{\mathcal{DT}g}$, $y_k^{\mathcal{T}^2g}$ or $y_k^{\mathcal{D}^2g}$) using y_k^g with different filter orders
3. Compute $\hat{n}$,$\hat{m}$ (resp. $\tilde{n}$, $\tilde{m}$) provided by the reassignment operators
4. **If** $\hat{n} \leq n$ (resp. $\tilde{n} \leq n$) **then** update TFR$[\hat{n},m]$ **otherwise** store the triplet $\left(y_k^g[n,m], \hat{n}, m\right)$ into a list
5. Update TFR$[n,m]$ using all previously stored triplets verifying $\hat{n} \leq n$ and remove them from the list

Numerical results

Resulting time-frequency representations



Signal reconstruction quality

Using the discrete-time version of Eq. (8) with $n_0 = t_0/T_s$.

(a)	n_0	8	18	26	**28**	30
	RQF (dB)	9.79	24.17	26.77	**26.82**	26.73
(b)	M	100	200	600	1000	**2400**
	RQF (dB)	20.56	24.90	29.48	30.50	**30.87**
(c)	μ	0.30	0.80	1.30	1.80	**2.30**
	RQF (dB)	20.83	27.28	29.68	30.35	**30.90**

Signal Reconstruction Quality Factor $\mathrm{RQF} = 10\log_{10}\left(\frac{\sum_n |x[n]|^2}{\sum_n |x[n]-\hat{x}[n]|^2}\right)$, of the recursive synchrosqueezed STFT computed for k = 5, L = 7 at SNR = 45 dB. Line (a), computed for M = 300, Line (b) and Line (c), computed for n0=28 and M=300.

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