A Constant-Gap Result on the Multi-Antenna Broadcast Channels with Linearly Precoded Rate Splitting

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Background and Motivations

We consider multi-antenna broadcast channel (BC) with independent private messages.

	dirty paper coding (DPC)	zero-forcing/MMSE
optimality	capacity achieving	max. degree of freedom (DoF)
complexity	non-linear, "hard"	linear, "easy"

Recall the progressive approximation of capacity:

$$\text{DoF} \ll \text{GDoF} \ll \text{Constant-gap} \ll \text{Capacity}$$

- DoF: treats all non-zero channels as equally strong
- GDoF: channel gains grow with the transmit power polynomially
- Constant-gap: no constraint on the channel gains

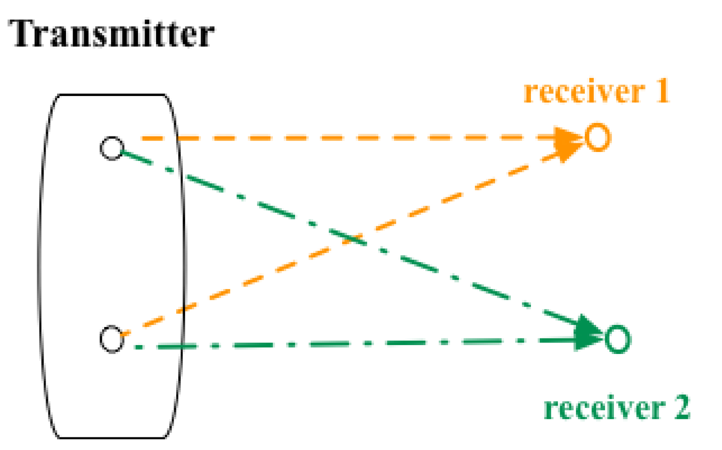
Main question: can linear tx schemes be constant-gap optimal?

Answer: yes, with rate splitting!

Rate-splitting: split the private messages to make part of it decodable.

System Model

Consider a two-user multi-input single-output (MISO) BC, with deterministic channel matrix $\boldsymbol{H}$



$$\begin{bmatrix} Y_1[t] \\ Y_2[t] \end{bmatrix} = \boldsymbol{H}\boldsymbol{x}[t] + \mathbf{Z}[t], \quad t = 1,\dots,n,$$

$$\boldsymbol{H} := [\boldsymbol{h}_1 \quad \boldsymbol{h}_2]^T$$

$$\frac{1}{n}\sum_{t=1}^{n} \|\boldsymbol{x}[t]\|^2 \le P$$

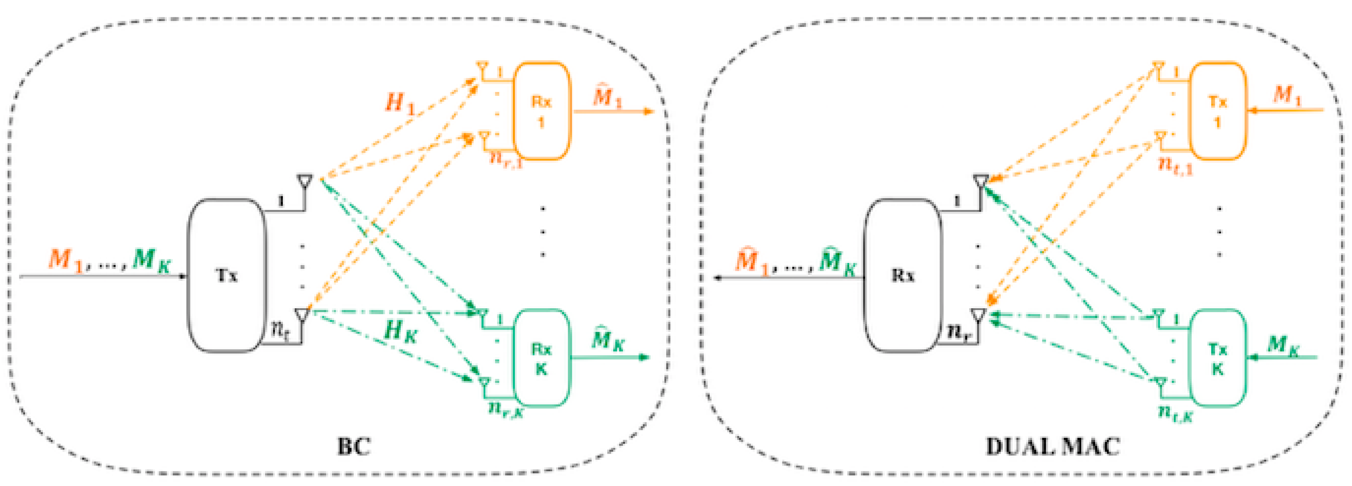
Sum Capacity characterized by dual MAC

$$C_{\text{sum}} = \max_{\text{tr}(\boldsymbol{\Lambda}) \le P} \log\det(\mathbf{I} + \boldsymbol{H}^H \boldsymbol{\Lambda} \boldsymbol{H})$$
$$\approx \log\left(1 + P\|\boldsymbol{h}_1\|^2 + P\|\boldsymbol{h}_2\|^2 + P^2 \det(\boldsymbol{H}\boldsymbol{H}^H)\right)$$

"≈" stands for constant-gap approximation

Recall the BC-MAC Duality

$$C_{\text{BC}}(\{\boldsymbol{H}_k\}_k, P) = \bigcup_{\{\boldsymbol{Q}_k\}_k:\ \sum_{k=1}^{K} \text{tr}(\boldsymbol{Q}_k) \le P} C_{\text{MAC}}(\{\boldsymbol{H}_k^H\}_k, \{\boldsymbol{Q}_k\}_k)$$



Linear Precoding with Private Streams is not Even GDoF Optimal

A pathological example channel matrix : $\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ f & g \end{bmatrix}$

$$C_{\text{sum}} \approx \max\left\{\log(1+P), \log(1+P|f|^2 + P|g|^2), \log(1+P^2|g|^2)\right\}$$

The achievable sum rate with linear precoding:

$$R_1 + R_2 \approx 2\min\left\{\frac{2}{|f|^2} + 2\frac{|g|^2}{|f|^2}\lambda_A,\ \lambda_A\right\} + 2|f|^2 + 2|g|^2\lambda_B$$

λ_A, λ_B is the larger singular value of $\mathbb{E}[\mathbf{X}_1\mathbf{X}_1^H]$ and $\mathbb{E}[\mathbf{X}_2\mathbf{X}_2^H]$ respectively.

$$\xrightarrow{f = P^{\alpha_f}, g = P^{\alpha_g}} \text{GDoF}: \quad d_{\text{LP}} \le 2 + 4\alpha_g - 2\alpha_f \quad \text{vs} \quad d_{\text{DPC}} = 2 + 2\alpha_g$$

$$d_{\text{LP}}(\alpha_f, \alpha_g) < d_{\text{DPC}}(\alpha_f, \alpha_g), \ \forall \alpha_f > \alpha_g > \alpha_f - \tfrac{1}{2} \ge 0$$

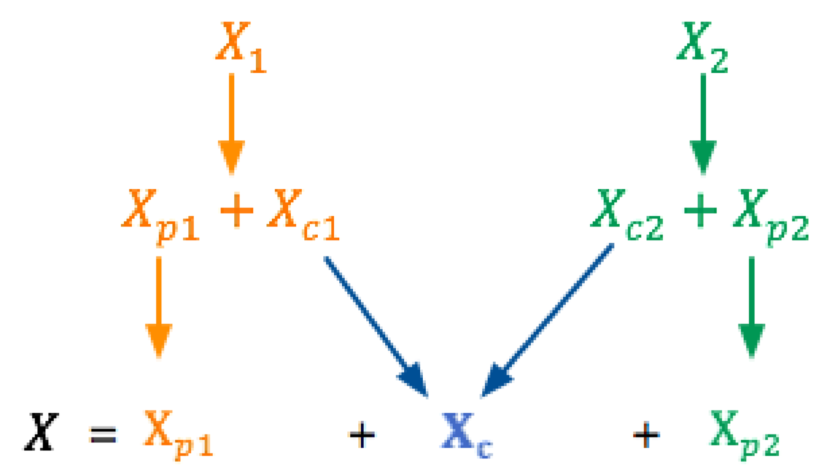
DPC	Linear Precodings
$\max\{\log(1+P\|\boldsymbol{h}_1\|^2), \log(1+P\|\boldsymbol{h}_2\|^2),$ $\log(1+P^2\det(\boldsymbol{H}\boldsymbol{H}^H))\}$	$\max\{\log(1+P\|\boldsymbol{h}_1\|^2), \log(1+P\|\boldsymbol{h}_2\|^2),$ $\log(1+\beta_\rho P^2\det(\boldsymbol{H}\boldsymbol{H}^H))\}$

$\beta_\rho := \min\left\{\frac{1-\rho^2}{\rho^2}, 1\right\}, \ \rho := \left|\frac{\boldsymbol{h}_1^H \boldsymbol{h}_2}{\|\boldsymbol{h}_1\|\|\boldsymbol{h}_2\|}\right|$

Introducing Rate Splitting

Each individual message is split into private and common parts that are encoded independently

- Common parts should be decodable by both receivers
- Private parts are treated as noise by the unintended receivers



Achievable Rate Region

- Each user decodes *jointly* its private stream and both common streams
- Equivalent to two MAC receivers, leading to the rate region:

$$R_{p1} \le \log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_1 \boldsymbol{h}_1^*\right)$$
$$R_{p2} \le \log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_2 \boldsymbol{h}_2^*\right)$$
$$R_{c1} + R_{c2} + R_{p1} \le \log\left(1 + \boldsymbol{h}_1^T (\boldsymbol{Q}_1 + \boldsymbol{Q}_0) \boldsymbol{h}_1^*\right)$$
$$R_{c1} + R_{c2} + R_{p2} \le \log\left(1 + \boldsymbol{h}_2^T (\boldsymbol{Q}_2 + \boldsymbol{Q}_0) \boldsymbol{h}_2^*\right)$$
$$R_{c1} + R_{c2} \le \min\left\{\log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_0 \boldsymbol{h}_1^*\right),\ \log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_0 \boldsymbol{h}_2^*\right)\right\}$$

$\mathbb{E}[\mathbf{X}_c\mathbf{X}_c^H] = \boldsymbol{Q}_0, \quad \mathbb{E}[\mathbf{X}_k\mathbf{X}_k^H] = \boldsymbol{Q}_k \ \ k = 1,2$

- Getting back to the individual rates $R_1 = R_{p1} + R_{c1}$ and $R_2 = R_{p2} + R_{c2}$: after the Fourier-Motzkin elimination:

$$R_1 \le \min\left\{\log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_1 \boldsymbol{h}_1^*\right) + \log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_0 \boldsymbol{h}_2^*\right),\ \log\left(1 + \boldsymbol{h}_1^T (\boldsymbol{Q}_1 + \boldsymbol{Q}_0) \boldsymbol{h}_1^*\right)\right\}$$
$$R_2 \le \min\left\{\log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_2 \boldsymbol{h}_2^*\right) + \log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_0 \boldsymbol{h}_1^*\right),\ \log\left(1 + \boldsymbol{h}_2^T (\boldsymbol{Q}_2 + \boldsymbol{Q}_0) \boldsymbol{h}_2^*\right)\right\}$$
$$R_1 + R_2 \le \min\left\{\log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_2 \boldsymbol{h}_2^*\right) + \log\left(1 + \boldsymbol{h}_1^T (\boldsymbol{Q}_1 + \boldsymbol{Q}_0) \boldsymbol{h}_1^*\right),\ \log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_1 \boldsymbol{h}_1^*\right) + \log\left(1 + \boldsymbol{h}_2^T (\boldsymbol{Q}_2 + \boldsymbol{Q}_0) \boldsymbol{h}_2^*\right)\right\}$$

Linearly Precoded Rate Splitting is Constant-Gap Optimal

- Use ZF for the private streams, with $P_1 = P_2 = \frac{P}{3}$:

$$\boldsymbol{Q}_1 = P_1\left(\mathbf{I} - \frac{\boldsymbol{h}_2^* \boldsymbol{h}_2^T}{\|\boldsymbol{h}_2\|^2}\right), \quad \boldsymbol{Q}_2 = P_2\left(\mathbf{I} - \frac{\boldsymbol{h}_1^* \boldsymbol{h}_1^T}{\|\boldsymbol{h}_1\|^2}\right).$$

- Isotropic precoding for the common stream: $\boldsymbol{Q}_0 = \frac{P}{3}\mathbf{I}_2$

$$R_1 + R_2 = \min\left\{\log\left(1 + \boldsymbol{h}_1^T \boldsymbol{Q}_1 \boldsymbol{h}_1^*\right) + \log\left(1 + \frac{P}{3}\|\boldsymbol{h}_2\|^2 + \boldsymbol{h}_2^T \boldsymbol{Q}_2 \boldsymbol{h}_2^*\right),\ \log\left(1 + \boldsymbol{h}_2^T \boldsymbol{Q}_2 \boldsymbol{h}_2^*\right) + \log\left(1 + \frac{P}{3}\|\boldsymbol{h}_1\|^2 + \boldsymbol{h}_1^T \boldsymbol{Q}_1 \boldsymbol{h}_1^*\right)\right\}$$
$$\ge \log\left(1 + P^2 \det(\boldsymbol{H}\boldsymbol{H}^H)\right) - \log 9$$

Using rate splitting with a simple power allocation, one can achieve the sum capacity to a constant gap for any channel realization.

Conclusions and Recent Progress

- Linear precoding with only private streams can have unbounded gap to the capacity.
- Rate splitting can help reduce the gap to a constant, i.e., it is constant-gap optimal.
- Constant-gap optimality is extended to the whole capacity region, for two-user MIMO BC.
- We have also proved that such optimality does not hold for more than two users.

More details can be found in:

- Z.Li and S. Yang "A Linearly Precoded Rate Splitting Approach and Its Optimality for MIMO Broadcast Channels," submitted to ITW2018. An extended version available soon.

Some references

- Weingarten et al., "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," TIT, Sept. 2006.
- T. Yoo and A. Goldsmith, "On the optimality of multiantenna broadcast scheduling using zero-forcing beamforming," JSAC, 2006.
- Han and Kobayashi, "A new achievable rate region for the interference channel," TIT, Jan. 1981.