Tree-Structured Quantization on Grassmann and Stiefel Manifolds

Data Compression Conference (virtual) 2021

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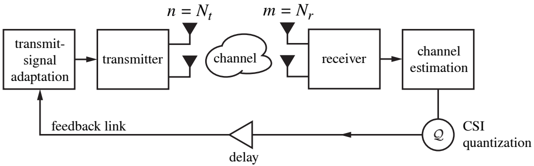
Motivation for Grassmann/Stiefel-Manifold Quantization



- Limited feedback MIMO single/multi-user wireless communications $\mathbf{H} \in \mathbb{C}^{n \times m}$, $m < n$

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}, \quad \mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}_m$$

- Grassmannian feedback: transmit-subspace $\mathrm{span}\,(\mathbf{H}) = \mathrm{span}\,(\mathbf{U})$
- Stiefel-manifold feedback: individual singular-vectors $\mathbf{U}$

Motivation for Grassmann/Stiefel-Manifold Quantization (II)

- Common approach for channels without structure (e.g., i.i.d. Rayleigh fading):
  Single-stage codebook-based quantization

$$\mathbf{Q}^* = \mathcal{Q}(\mathbf{H}) = \underset{\mathbf{Q}_\ell \in \mathcal{Q}_m^{(n)}}{\arg\min}\ \mathrm{d}(\mathbf{H}, \mathbf{Q}_\ell)$$

- Grassmannian and Stiefel-manifold quantization metrics

$$\mathrm{d}(\mathbf{H}, \mathbf{Q}_\ell) = \mathrm{d}_\mathrm{c}^2(\mathbf{U}, \mathbf{Q}_\ell) = m - \mathrm{tr}\left(\mathbf{U}^\mathrm{H}\mathbf{Q}_\ell\mathbf{Q}_\ell^\mathrm{H}\mathbf{U}\right),$$

$$\mathrm{d}(\mathbf{H}, \mathbf{Q}_\ell) = \|\mathbf{U} - \mathbf{Q}_\ell\|_F^2 = 2m - 2\,\Re\left(\mathrm{tr}\left(\mathbf{U}^\mathrm{H}\mathbf{Q}_\ell\right)\right)$$

- Single-stage quantization codebooks

$$\mathcal{Q}_m^{(n)} = \left\{\mathbf{Q}_\ell \in \mathbb{C}^{n\times m} | \mathbf{Q}_\ell^\mathrm{H}\mathbf{Q}_\ell = \mathbf{I}_m, \ell \in \{1, \ldots, K\}\right\}$$

- Main issue: codebook size has to grow exponentially with dimensions $n, m \Rightarrow$ complexity explodes

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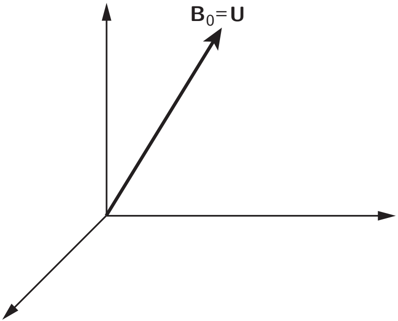
Recursive (Greedy) Multi-Stage Grassmannian Quantization – Toy Example $n \times m = 3 \times 1$



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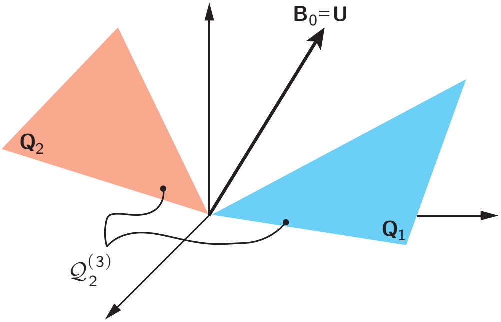

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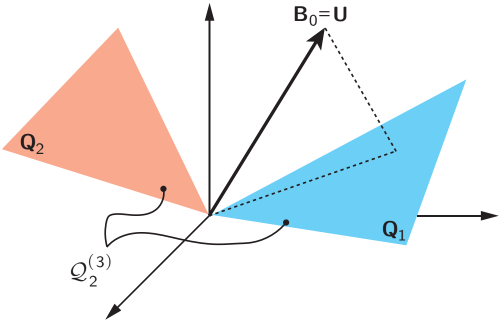

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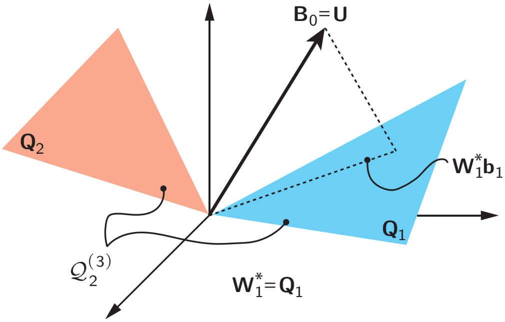

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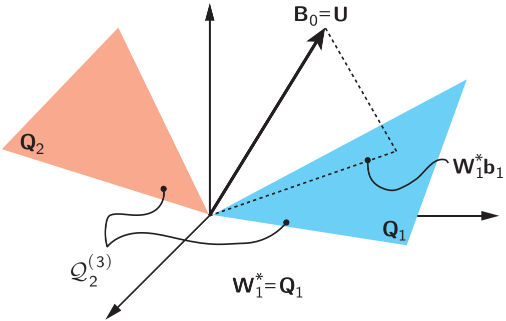

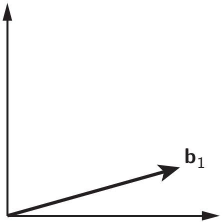

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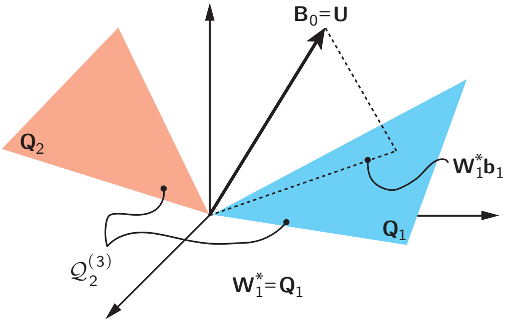

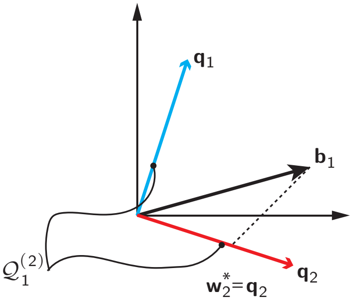

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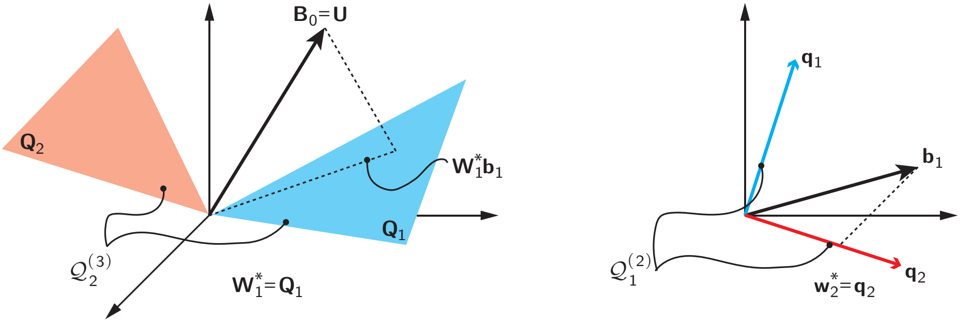

$\mathbf{Q}^* = \mathbf{W}_1^* \mathbf{w}_2^*$

Recursive (Greedy) Multi-Stage Grassmannian Quantization

- Denote the initial quantizer input as $\mathbf{B}_0 = \mathbf{U} \in \mathbb{C}^{n \times m}$ with $d_0 = n$
- Recursive (greedy) multi-stage quantization in R stages $i \in \{1, \dots, R\}$
- Input of stage i: $\mathbf{B}_{i-1} \in \mathbb{C}^{d_{i-1} \times m}$
- Output of stage i: quantized subspace $\mathbf{W}_i^* \in \mathbb{C}^{d_{i-1} \times d_i}$ and combining matrix $\mathbf{B}_i \in \mathbb{C}^{d_i \times m}$
- Dimension shrinking: $d_{i-1} > d_i$, $d_0 = n$, $d_R = m$

$$\mathbf{W}_i^* = \underset{\mathbf{Q} \in \mathcal{Q}_{d_i}^{(d_{i-1})}}{\arg\min} \; \mathrm{d}_{\mathrm{c}}^2 \left(\mathbf{B}_{i-1}, \mathbf{Q} \right),$$

$$\mathbf{B}_i = \mathbf{W}_i^{*\mathrm{H}} \mathbf{B}_{i-1} \left(\mathbf{B}_{i-1}^{\mathrm{H}} \mathbf{W}_i^* \mathbf{W}_i^{*\mathrm{H}} \mathbf{B}_{i-1} \right)^{-\frac{1}{2}}$$

- Subspace quantization-based combining (SQBC) matrix $\mathbf{B}_i$: $\mathrm{d}_{\mathrm{c}}^2 \left(\mathbf{B}_{i-1}, \mathbf{W}_i^* \right) = \mathrm{d}_{\mathrm{c}}^2 \left(\mathbf{B}_{i-1}, \mathbf{W}_i^* \mathbf{B}_i \right)$
- Final quantizer output: $\mathbf{Q}^* = \prod_{i=1}^{R} \mathbf{W}_i^* \in \mathbb{C}^{n \times m}$

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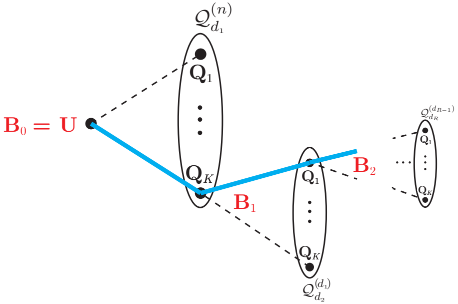
Quantizer Structures



- Greedy, tree-structured and trellis quantization

S. Schwarz, "Non-coherent broadcasting based on Grassmannian superposition transmission," in *IEEE International Symposium on Information Theory (ISIT)*, July 2021, pp. 1–5, under review

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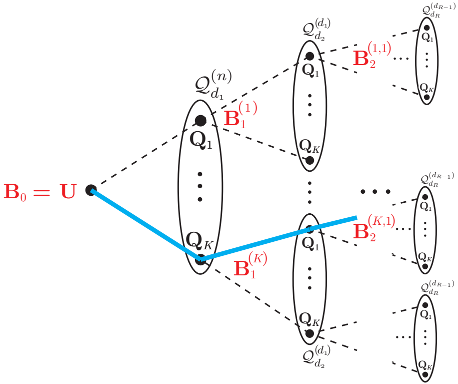

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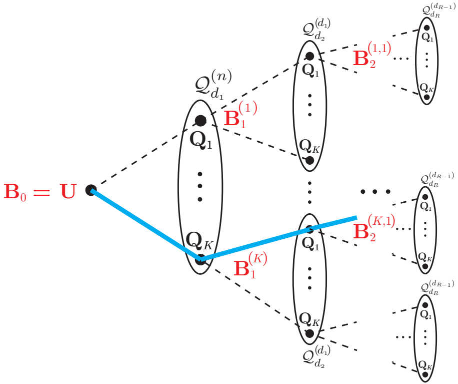

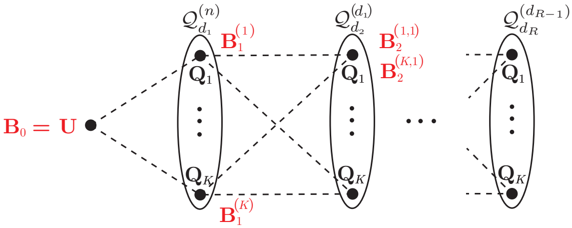

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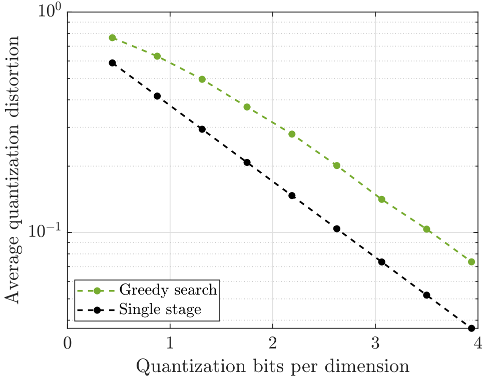
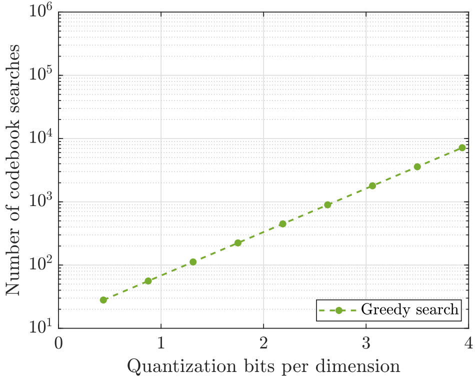
Simulation Setup

- Quantization of points uniformly distributed on the Grassmannian of dimensions $n = 16$ and $m = 2$
- $R = 14$ quantization stages with dimension step-size $d_{i-1} - d_i = 1$
- Quantization distortion as a function of quantization bits per dimension $\bar{b} = b/(nm)$ with $b \in [14, 126]$
- Random vector quantization codebooks
- In the paper: similar results for the Stiefel-manifold

Greedy, Tree and Trellis Quantization





- Complexity of single stage: 1.6e4 to 8.5e37
- Tree-search of increasing search-depth (by far not up to full tree-search)
- Full and pruned trellis-search

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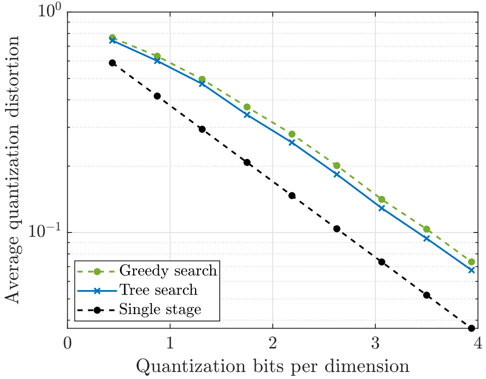

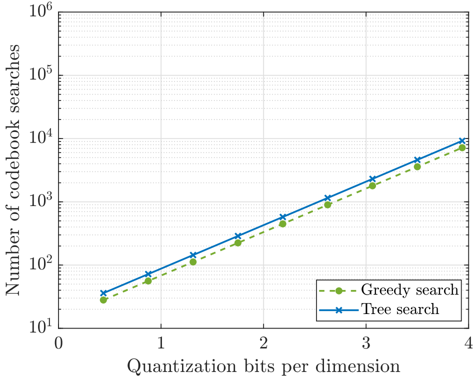

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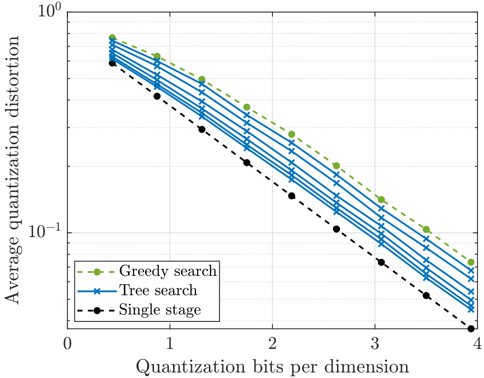

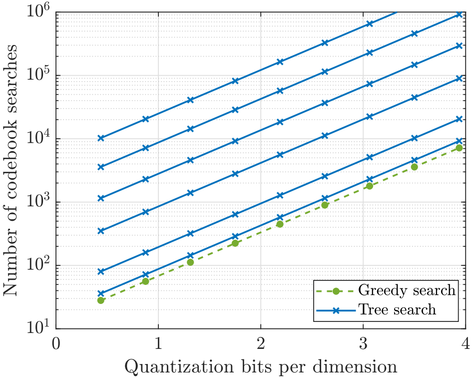

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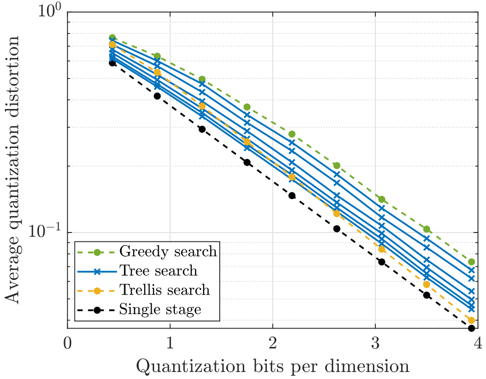

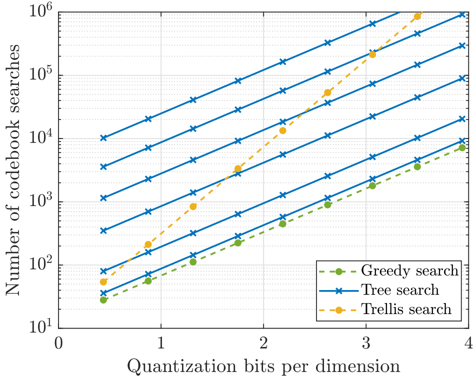

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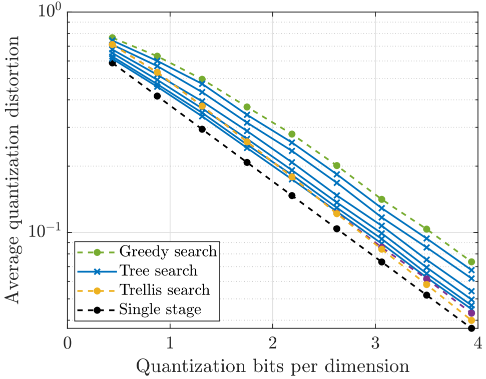

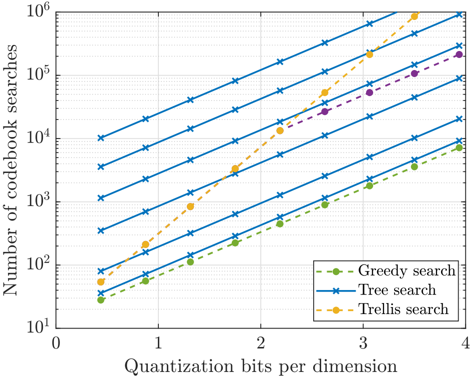

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Conclusions and Future Work

- Recursive multi-stage quantization can support high-resolution with acceptable complexity and a small to moderate performance loss compared to single-stage quantization
- For Grassmannian quantization we provide theoretical rate-distortion results in the paper (assuming random vector quantization)
- Todo: derive similar theoretical results for quantization on the Stiefel manifold

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