On Universal Codes for Integers

Wallace Tree, Elias Omega and Beyond

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... Australia

Motivation

- Officially:

  compress integers

  inductive inference $\{H_1, H_2, ...\}, \ \Pr(H_i) = \Pr(i) = 2^{-|\mathsf{CW}(i)|}$

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- Actually:

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- * (i) would like $\Pr(i)$ and $\Pr(i+1)$ to be similar,
  (ii) sometimes want $\Pr(n) \sim 1/n$ (and pdf$(x) \sim 1/x$), giving $|\mathsf{CW}(n)| \sim \log(n)$; impossible of course.

... a fancy

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$$\int_1^\infty \frac{1}{x^{1+\delta}}, \quad \int_2^\infty \frac{1}{x(\log x)^{1+\delta}}, \quad ..., \quad \text{are finite,}$$

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$\int\limits_1^\infty \frac{1}{x^{1+\delta}}, \quad \int\limits_2^\infty \frac{1}{x(\log x)^{1+\delta}}, \quad \ldots,$ are finite,

and $\log(x(\log x)^{1+\delta})) = \log x + (1+\delta) \log \log x$

Plan

Some Universal Codes for Integers:

omega (i.e., Elias ω)

omega$_p(s)$

  omega^2 = omega$_p$(omega)

omega$_r(t)$

  omega* = omega$_r$(omega)

? Is there an ultimate code ?

WTC (Wallace's tree-based code)

Comparisons

Elias omega (ω) (1975)

n=1	"1"
2	"010"
3	"011"
4	"00100"
...	...

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$$1\ 0\ 0\ 1\ 0\ 0 \quad = 36, \quad \text{len}=6$$

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- e.g., n = 36

$$\begin{array}{lll} \quad\quad 1\ 0\ 0\ 1\ 0\ 0 & = 36, & \text{len}=6 \\ 1\ 0\ 1 & = 5, & \text{len}=3 \end{array}$$

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n=1	"1"
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$$
\begin{array}{llll}
 & & 1\ 0\ 0\ 1\ 0\ 0 & = 36, \quad \text{len}=6 \\
 & 1\ 0\ 1 & & = 5, \quad \text{len}=3 \\
1\ 0 & & & = 2, \quad \text{len}=2
\end{array}
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n	
n=1	"1"
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- e.g., n = 36

1 0 0 1 0 0	= 36,	len=6
1 0 1	= 5,	len=3
1 0	= 2,	len=2
1	= 1,	len=1

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 & 1 & = 1, & \text{len}=1
\end{array}
$$

- 1 10 101 100100 use leading bits for #sections...

  $\underline{0}$ $\underline{0}0$ $\underline{0}01$ $\underline{1}00100$ note, $\underline{0001}$ is unary code for 4

"000001100100"

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advocated by J. Rissanen (1983) for use in inference,
can be seen as an approximation to omega.

omega has a big step when a new section kicks in, e.g.,
15 $\rightarrow$ 0 01 1111 (7 bits); 16 $\rightarrow$ 0 00 000 10000 (11 bits).

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and omega$^2 =$ omega$_p$(omega).

Even a "recursive" version, omega$_r(t)$ that uses itself for $\#$sections ...

$\text{omega}_r(t)$, omega* ...

```
function omega_r_enc(t)
 { function enc(n)
    { var todo = n, CW = "", ...;
      for( nTet = 1; ; nTet ++ )
       { omega's logic but trim off the
         leading bit of each section;

         if( nSections == 1 ) break;
         todo = nSections - 1;      // !
       }//nTet
      return t(nTet) ++ CW;         // !
    }//enc
   return enc;
 }//omega_r_enc

function omega_star_enc(n) = omega_r_enc(omega_enc)(n)
```

... omega*

e.g., n=36

trim(omega(36)) ~~0~~ ~~0~~0 ~~0~~01 ~~1~~00100

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trim(omega(36)) $\not{0}\ \not{0}0\ \not{0}01\ \not{1}00100$

trim(omega(s−1=3)) $\not{0}\ \not{1}1$

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e.g., n=36

trim(omega(36)) $\not 0\ \not 0 0\ \not 0 01\ \not 1 00100$

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omega(#t=3)	0 11			
	0 11	_	1	00100100

"011100100100"

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  $\mathsf{CW}'(4) = \mathsf{CW}(2) + +"1"$

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  ...
  $\mathsf{CW}'(2n-1) = \mathsf{CW}(n) ++ "0"$
  $\mathsf{CW}'(2n) = \mathsf{CW}(n) ++ "1"$
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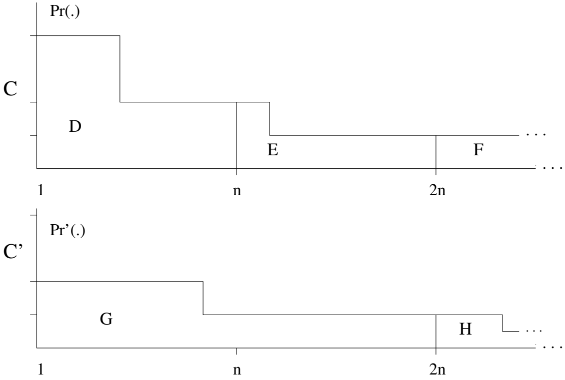
  $\sum_{n\geq 1} \mathrm{Pr}'(n) = 2\sum_{n\geq 1} \mathrm{Pr}(n)/2 = 1.$

... continuous improvement.



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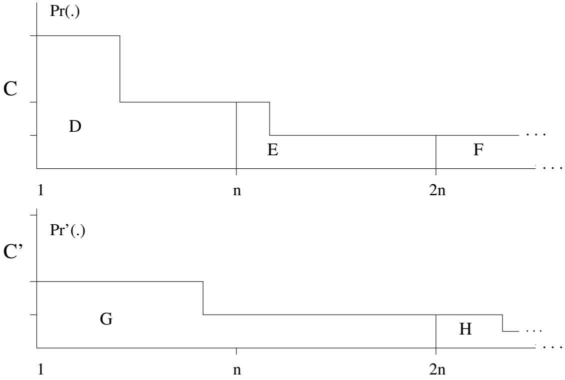

- $D + E + F = G + H = 1,$

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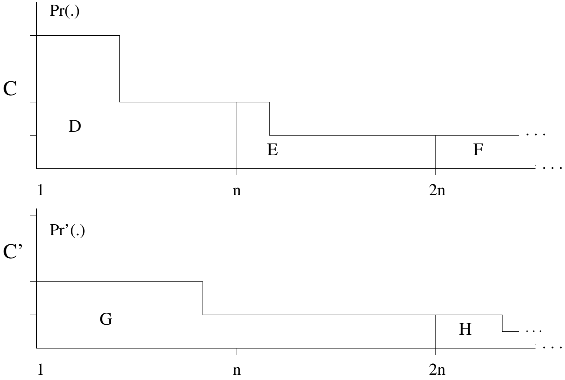

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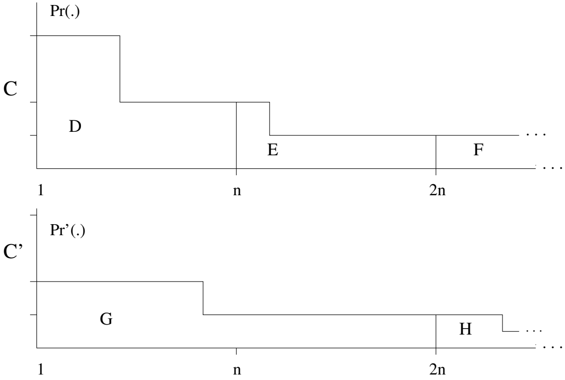

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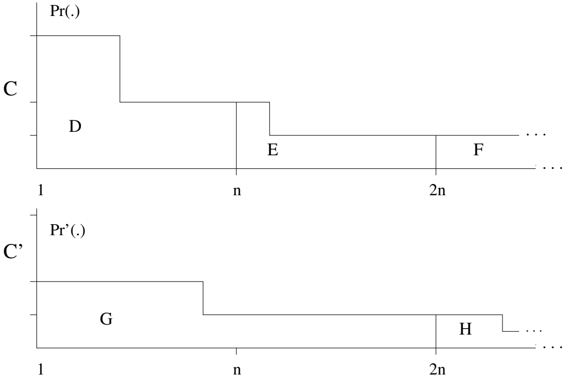

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  so $H > F$.
- **C'** is "better" than **C**, *asymptotically*.

WTC ...

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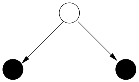
0:	1:	"0"	
1:	2:	"100"	

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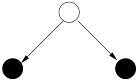
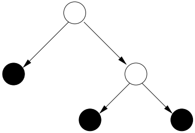
1:	2:	"100"	
2:	3:	"10100"	

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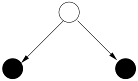
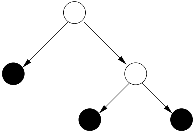
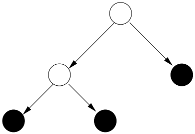
1:	2:	"100"	
2:	3:	"10100"	
3:	4:	"11000"	

...

... WTC, Catalans, paths

r↑								
5	1	5	14	28	42	42	0	...
4	1	4	9	14	14	0		
3	1	3	5	5	0			
2	1	2	2	0				
1	1	1	0					
0	1	0						
	0	1	2	3	4	5		→c

$paths$

$paths_{r,c} = \#$ Dyck paths from (r, c) to $(-1, 0)$.
$\leftarrow$ "1", $\downarrow$ "0".

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$(paths_{f,f} = C_f.)$

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WTC0(35) = WTC1(36) = "10111010000"

... WTC complexity

- A l o n g compare and (for a "1") subtract give one bit of code, so provided paths$_{r,c}$ and the cummulative Catalans are cached and we have a "binary" representation of n, encode and decode take $O(\log(n))^2)$ time.

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- Is there a faster algorithm?
- There is a fast approximation for $|\text{CW}(n)|$: $\log_2(n) + 1.5\log_2(\log_2(n)) + c$.

Comparisons [www]

n	Fibonacci	ω	ω^*	WTC
1	2	<u>1</u>	<u>1</u>	<u>1</u>
2	<u>3</u>	<u>3</u>	4	<u>3</u>
3	4	<u>3</u>	4	5
4	<u>4</u>	6	7	5
...	...	...	...	...
10^6	30	31	36	<u>27</u>
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(|CW(n)|)

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ω beats WTC permanently from ??? on.
ω^* beats ω from $n = 2 \uparrow\uparrow 8$ on.

Conclusions

- omega^2 and omega* are fun but not better in practice at compressing typical (ha, tiny!) integers.
- WTC compresses almost all small and "quite large" integers more than omega.
- WTC is nicer than omega and log* for inference,

  a bit slow for "data compression" of integers but

  fast enough for inference and

  there is a fast approximation for $|\mathsf{CW}(n)|$.

Interesting

- A. V. Levenstein, "*in Russian*", 1968, see D. Salomon, *Variable-length Codes for Data Compression*, Springer, p.80, 2007.
- F. Ruskey, *Generating balanced parenthesis strings by prefix shifts*, in CATS, Wollongong, pp.107-115, 2008.
- C. S. Wallace, *Statistical and Inductive Inference by Minimum Message Length*, Springer, 2005.

End