ON THE NULL SPACE CONSTANT FOR LP MINIMIZATION

In IEEE Signal Processing Letters, vol. 22, no. 10, pp. 1600-1603, 2015.

Laming Chen and Yuantao Gu (http://gu.ee.tsinghua.edu.cn/)
Department of Electronic Engineering, Tsinghua University, Beijing 100084, China
Email: chen-lm06@mails.tsinghua.edu.cn; gyt@tsinghua.edu.cn

1. Background

- **ℓ_p-minimization $(0 \le p \le 1)$ in sparse recovery**

$$\underset{\mathbf{x}}{\operatorname{argmin}} \sum_{i=1}^{N} |x_i|^p \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{y}$$

  - **Fundamental optimization problems**
  - **Performance can be viewed as benchmark**
- **Null space constant**

$$\gamma(\ell_p, \mathbf{A}, k) = \max_{\#S \le k} \sup_{\mathbf{z} \in \mathcal{N}(\mathbf{A})} \frac{\sum_{i \in S} |z_i|^p}{\sum_{i \notin S} |z_i|^p}$$

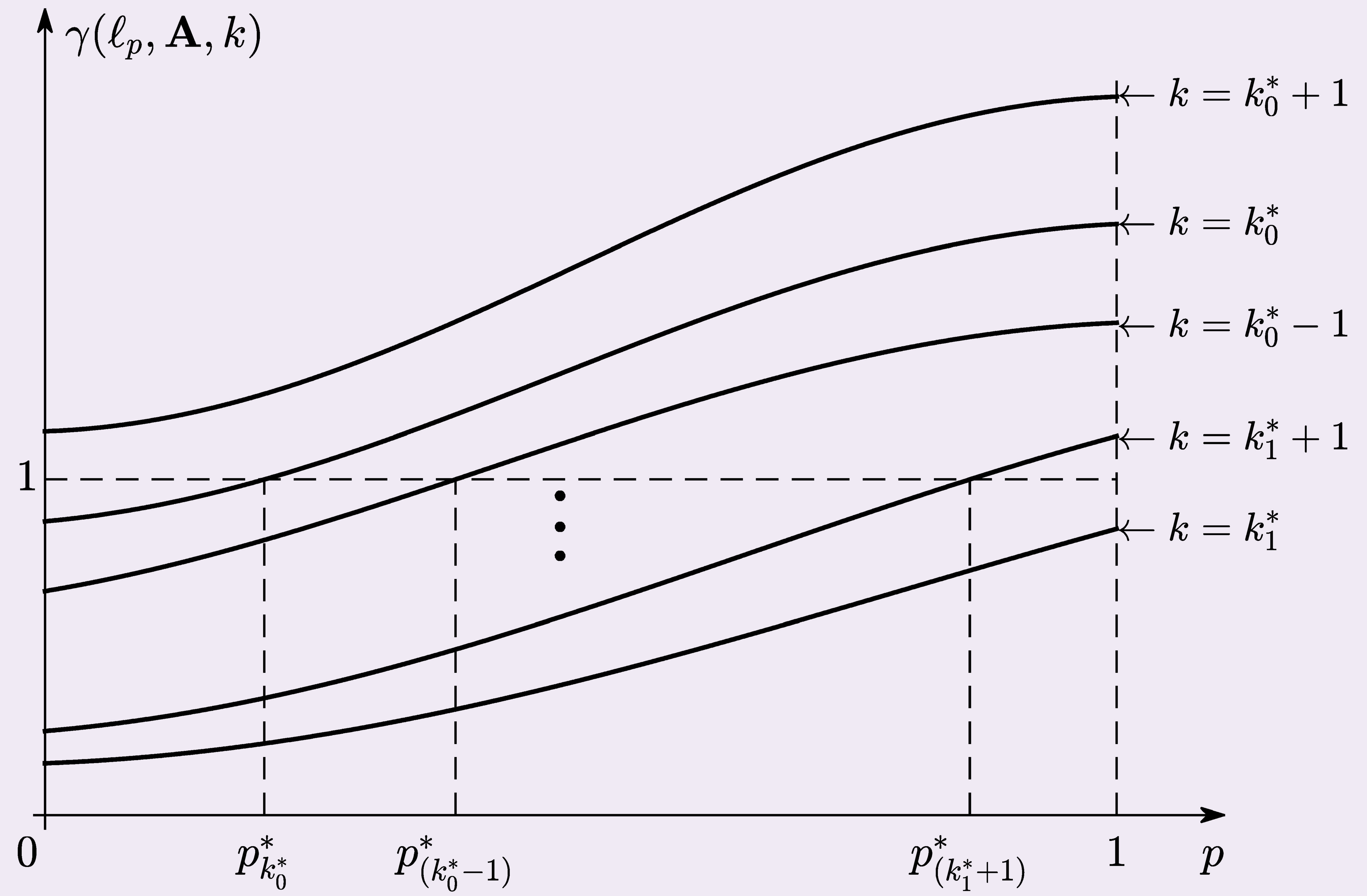
  - $\gamma(\ell_p, \mathbf{A}, k) < 1 \Leftrightarrow \ell_p$**-minimization can recover any k-sparse signal**
- **Current results**
  - $\gamma(\ell_p, \mathbf{A}, k)$ **is non-decreasing in k**
  - $\gamma(\ell_p, \mathbf{A}, k)$ **is non-decreasing in $p \in [0,1]$**

2. Results for General $\mathbf{A}$

- **Let $\mathrm{Spark}(\mathbf{A})$ denote the smallest number of columns from $\mathbf{A}$ that are linearly dependent.**
- **Theorem 1. Suppose $\mathrm{Spark}(\mathbf{A}) = L + 1$. Then for $p \in [0,1]$, $\gamma(\ell_p, \mathbf{A}, k)$ is strictly increasing in k when $k \le L$.**
- **Theorem 2. Suppose $\mathrm{Spark}(\mathbf{A}) = L + 1$. Then for $k \le L$, $\gamma(\ell_p, \mathbf{A}, k)$ is continuous in p when $p \in [0,1]$.**
- **Corollary 1. If $\gamma(\ell_0, \mathbf{A}, k) < 1$, then there exists a $p > 0$ such that $\gamma(\ell_p, \mathbf{A}, k) < 1$.**
- **Corollary 2. If $\gamma(\ell_0, \mathbf{A}, k) < 1$ and $\gamma(\ell_1, \mathbf{A}, k) \ge 1$, then there exists a $p_k^* > 0$ such that $\gamma(\ell_p, \mathbf{A}, k) < 1$ if and only if $p \in [0, p_k^*)$.**
- **Corollary 3. Let k_p^* denote the largest integer such that $\gamma(\ell_p, \mathbf{A}, k_p^*) < 1$. Then k_p^* is a non-increasing step function in $p \in [0,1]$, and the height of each step is 1.**

3. Results for Random $\mathbf{A}$

- **Theorem 3. Suppose the entries of $\mathbf{A} \in \mathbb{R}^{M \times N}$ are i.i.d. and satisfy a continuous probability distribution. Then for $k \le M$, $\gamma(\ell_p, \mathbf{A}, k)$ is strictly increasing in $p \in [0,1]$ with probability 1.**



4. Conclusion

- **Some basic properties of $\gamma(\ell_p, \mathbf{A}, k)$**
  - $\gamma(\ell_p, \mathbf{A}, k)$ **is strictly increasing in k**
  - $\gamma(\ell_p, \mathbf{A}, k)$ **is continuous in p**
  - $\gamma(\ell_p, \mathbf{A}, k)$ **is strictly increasing in p with probability 1 for random $\mathbf{A}$**
- **Possible future works include the properties of $\gamma(\ell_p, \mathbf{A}, k)$ in $\mathbf{A}$.**