Resolution Limits of 20 Questions Search Strategies for Moving Targets

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Joint work with Lin Zhou (Beihang University)

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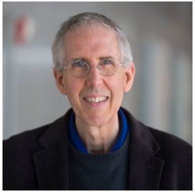

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Jun. 9, 2021

Origin: 20 Questions Game

Responder

Questioner

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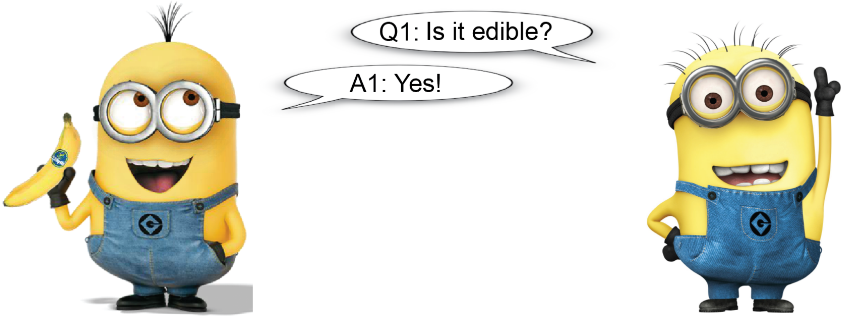

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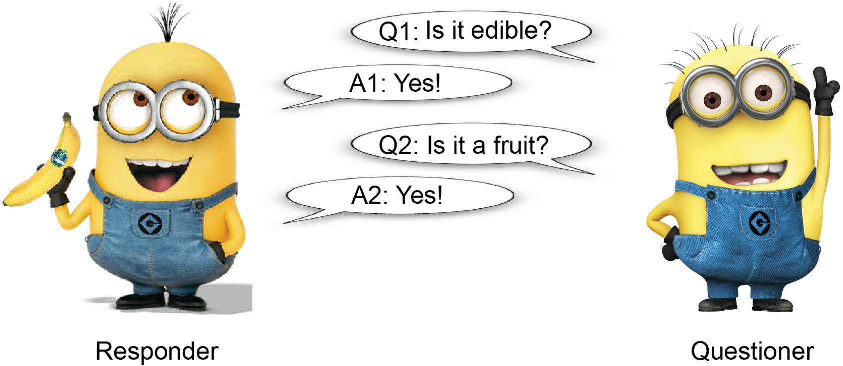

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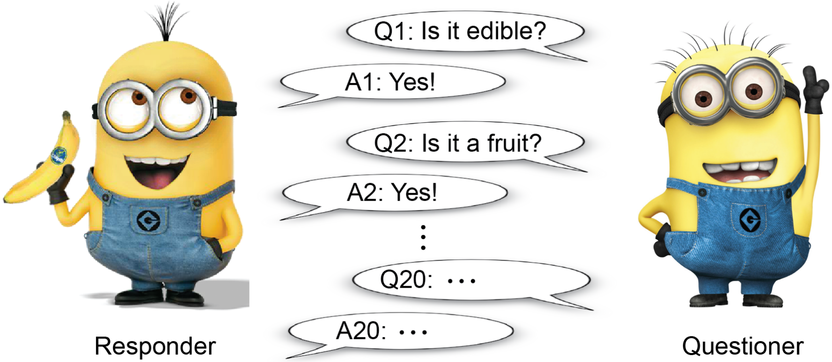

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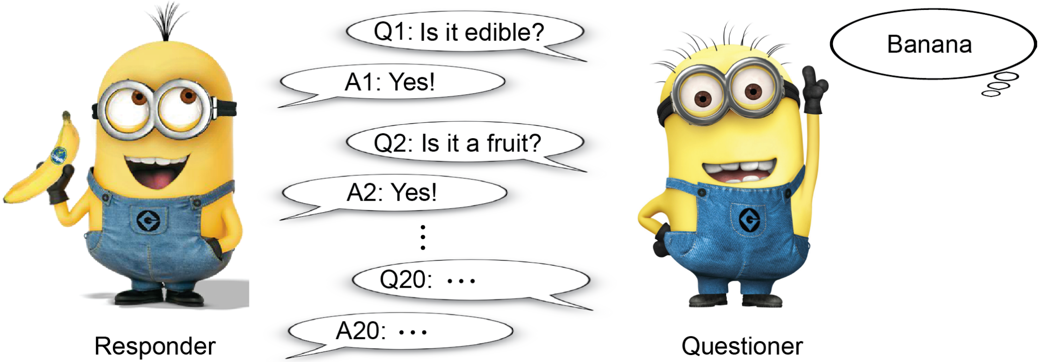
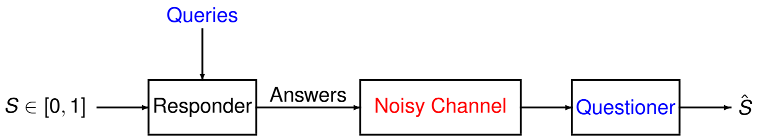


Recap: Search with Noise for a Stationary Target



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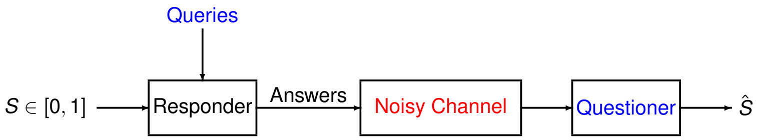

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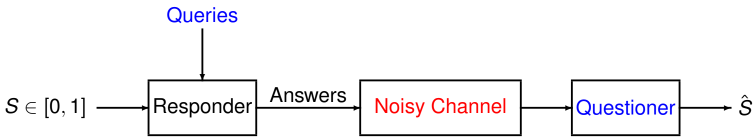

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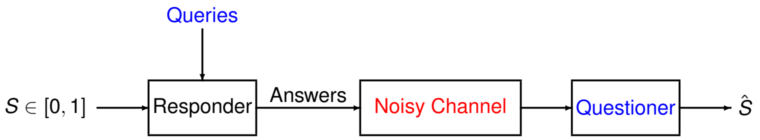

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- Target: estimate a random variable S with unknown distribution
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- Motivation: diverse applications including
  - medical diagnosis, chemical triage, human-in-the-loop decision-making
  - fault-tolerant communications, beamforming design in millimeter wave communication
  - target localization with a sensor network, object localization in an image

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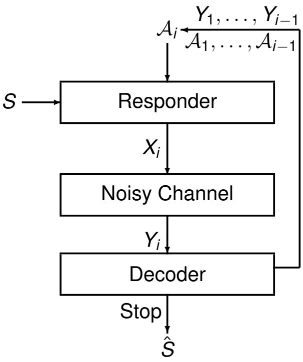

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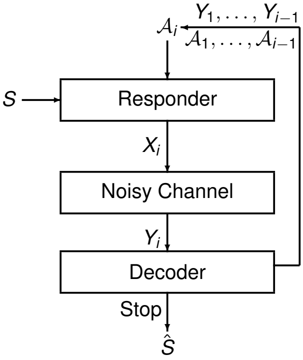
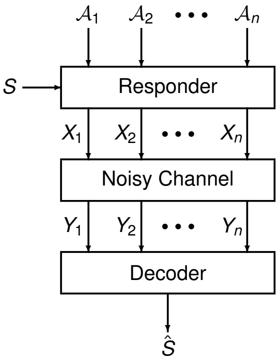


Non-adaptive



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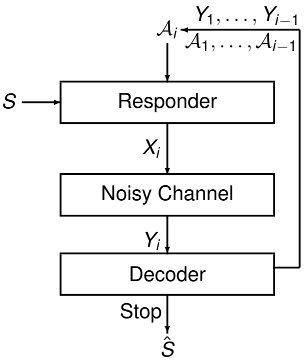

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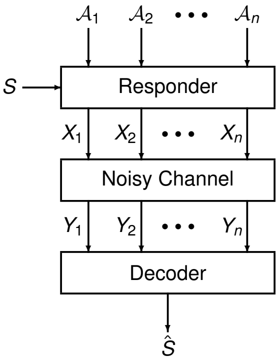
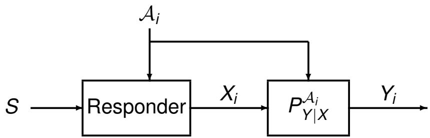


Measurement-Dependent[1] Noise Model



- Given a query (measurement) $\mathcal{A}$, a responder's noiseless answer is corrupted by measurement-dependent noise via $P^{\mathcal{A}}_{Y|X}$.

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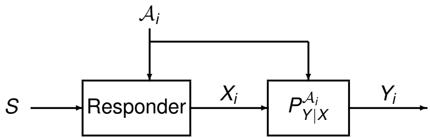

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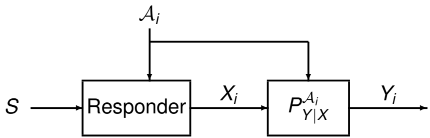

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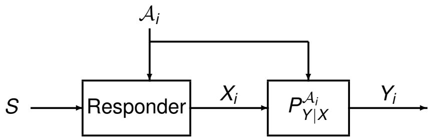

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- For any $(q_1, q_2) \in [0, 1]^2$, $|f(q_1) - f(q_2)| \leq \mu |q_1 - q_2|$.
- When f is a constant value function, the noise model reduces to a measurement-independent model.

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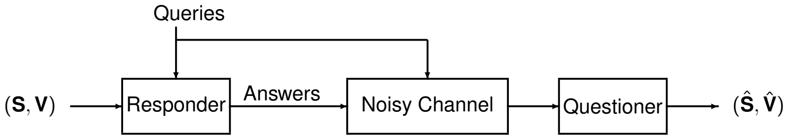
Problem Formulation: Search for a Multidimensional Moving Target



- Target: estimate the trajectory of a *d*-dimensional moving target with initial location $\mathbf{S} = (S_1, \ldots, S_d) \in [0, 1]^2$ and moving velocity $\mathbf{V} = [V_1, \ldots, V_d] \in [-v_+, v_+]^d$

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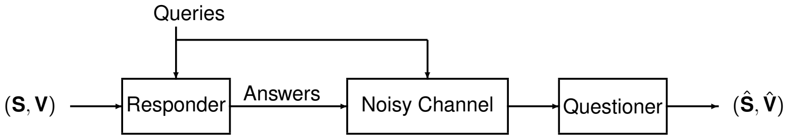

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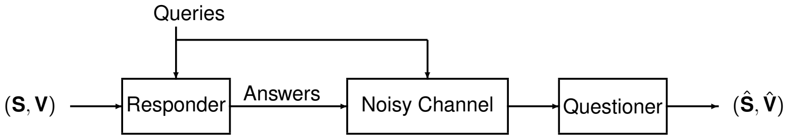

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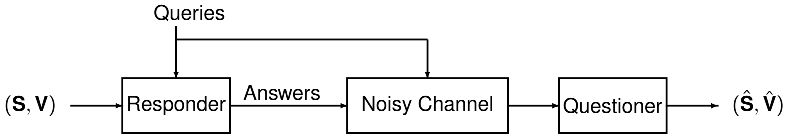

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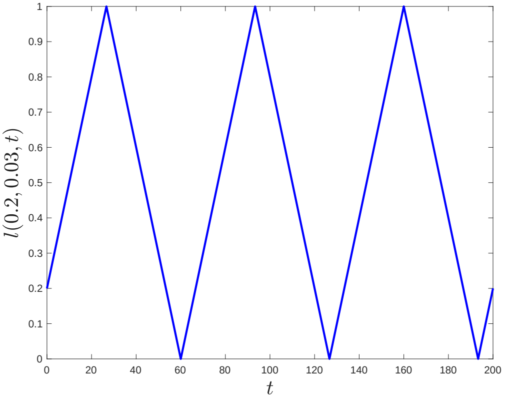
- Applications: search for a moving target (e.g., a car, wild animals, missing airplane) using sensor networks or satellites

The Torus model for the Moving Target

Given initial location $\mathbf{s} = (s_1, \ldots, s_d)$ and moving velocity $\mathbf{v} = (v_1, \ldots, v_d)$, at each time $t \in \mathbb{R}_+$, the real time location of the target at i-th dimension satisfies:

$$\mathrm{l}(s_i, v_i, t) := \begin{cases} 1 & \text{if } \mathrm{mod}(s_i + tv_i, 2) = 1, \\ s_i + tv_i - \lfloor s_i + tv_i \rfloor & \text{if } s_i + tv_i \in \bigcup_{h \in \mathbb{N}} [2h, 2h+1), \\ \lceil s_i + tv_i \rceil - (s_i + tv_i) & \text{otherwise}, \end{cases}$$



Definition of Non-Adaptive Query Procedures

Given any $(n, d) \in \mathbb{N}^2$, $\delta \in \mathbb{R}_+$ and $\varepsilon \in [0, 1)$, a $(n, d, \delta, \varepsilon)$-non-adaptive query procedure consists of

- n queries $\mathcal{A}^n$ where at time i, questioner asks whether the moving target's current location lies in set $\mathcal{A}_i \subset [0, 1]^d$
- and a decoder $g : \mathcal{Y}^n \to [0, 1]^d \times \mathcal{V}^d$ such that the worst-case excess-resolution probability satisfies

$$\mathrm{P_e}(n, d, \delta) := \sup_{f_{\mathbf{SV}}} \Pr\left\{ \max_{t \in [0:n]} \| l(\hat{\mathbf{S}}, \hat{\mathbf{V}}, t) - l(\mathbf{S}, \mathbf{V}, t) \|_\infty > \delta \right\} \leq \varepsilon.$$

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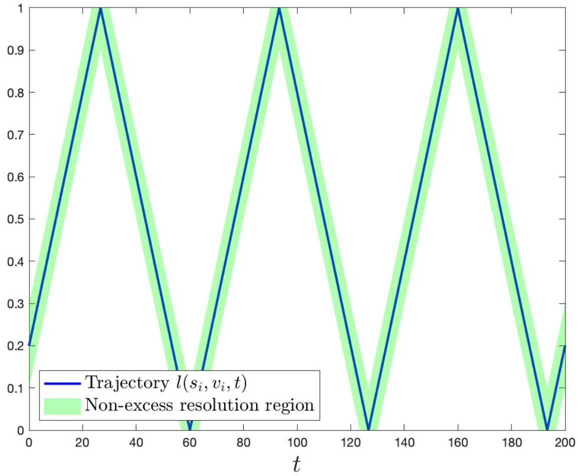
  - Excess-resolution event won't occur if $|\hat{S}_i - S_i| \leq \alpha\delta$ and $n|\hat{V}_i - V_i| \leq (1 - \alpha)\delta$ for all dimensions $i \in [d]$

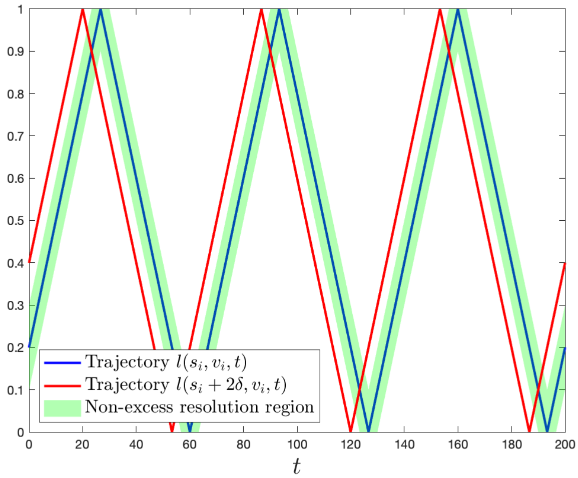
Explanation of the Non-Excess Resolution Event

For each $i \in [d]$, the i-th dimension does *not* incur excess-resolution if the estimated trajectories are within δ around the true trajectory at each time (in green shaded region).



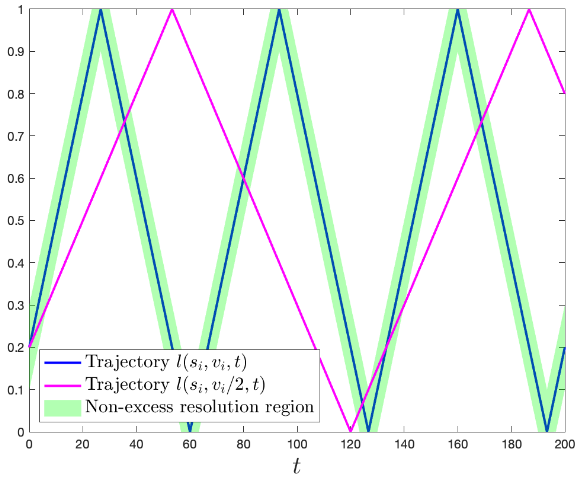
Excess-Resolution Case 1: Wrong Estimate of Initial Location

If the initial location s_i is estimated wrongly such that $|\hat{s}_i - s_i| > \delta$, then an excess-resolution event occurs.



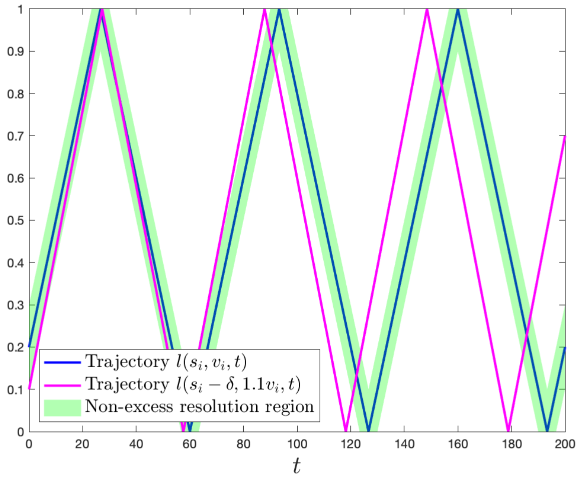
Excess-Resolution Case 2: Wrong Estimate of the Velocity

If the velocity v_i is estimated wrongly such that $n|\hat{v}_i - v_i| > 2\delta$, then an excess-resolution event occurs.



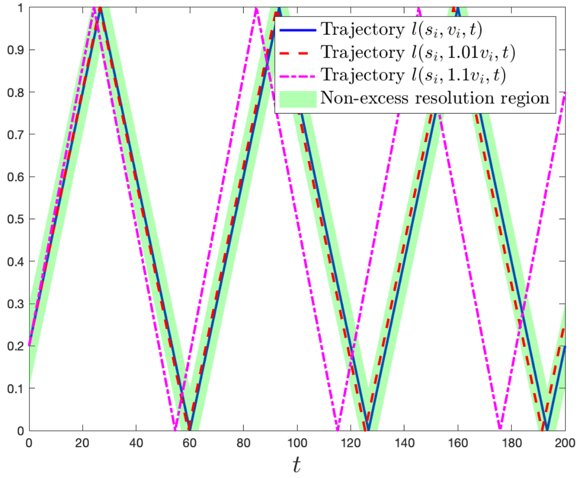
Excess-Resolution Case 3: Wrong Estimate of Both

If both the location s_i and the velocity v_i are estimated wrongly, an excess-resolution event occurs.



Excess-Resolution Event and the Number of Queries

Longer search time *n* requires more accurate estimation of the initial location and the velocity



Fundamental Limit

- Given any number of queries $n \in \mathbb{N}$ and $\varepsilon \in [0,1]$,

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- Dual quantity (sample complexity):

$$n^*(d, \delta, \varepsilon) = \inf\{n \in \mathbb{N} : \delta^*(n, d, \varepsilon) \leq \delta\}$$

Characterization of the Minimal Achievable Resolution $\delta^*(n, d, \varepsilon)$

Theorem 1

For any $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, the minimal achievable resolution $\delta^*(n,d,\varepsilon)$ satisfies

- *if $nv_+ = O(n^t)$ for $t \in [0.5, 1)$,*

$$-2d \log \delta^*(n,d,\varepsilon) = nC + O(nv_+);$$

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Characterization of the Minimal Achievable Resolution $\delta^*(n, d, \varepsilon)$

Theorem 1

For any $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, the minimal achievable resolution $\delta^*(n,d,\varepsilon)$ satisfies

- *if $nv_+ = O(n^t)$ for $t \in [0.5, 1)$,*

$$-2d \log \delta^*(n,d,\varepsilon) = nC + O(nv_+);$$

- *if $nv_+ = O(n^t)$ for $t \in [0, 0.5)$*

$$-2d \log \delta^*(n,d,\varepsilon) = nC + \sqrt{n\mathrm{V}_\varepsilon}\Phi^{-1}(\varepsilon) + O(\max\{nv_+, \log n\});$$

- Provides tight approximation to the performance of optimal non-adaptive queries
- The two assumptions on maximal speed v_+ specify two regimes
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  - Regime 2: # queries is fewer than $O(1/v_+^2)$ (slow target, $v_+ = o(\frac{1}{\sqrt{n}})$)
- In Regime 2, the first-order asymptotic result is *not* sufficient (see next slide)

First- and Second-order Asymptotics for the Resolution Decay Rate

- First-order asymptotics: the asymptotic resolution decay rate

$$\lim_{n\to\infty} \frac{-\log \delta^*(n,d,\varepsilon)}{n} = \frac{C}{2d}$$

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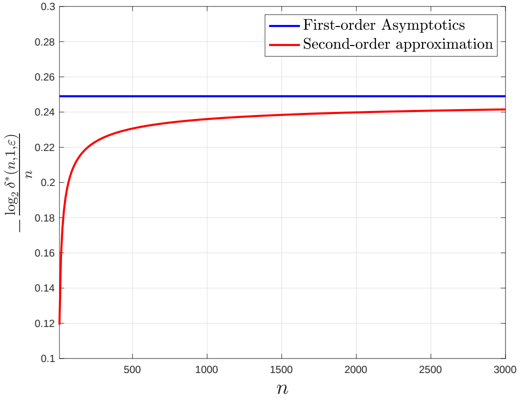
- Second-order asymptotics: characterize the backoff from first-order ($nv_+ = o(\sqrt{n})$)



Further Remarks

- Refines the result by Kaspi *et al.*, TIT 2018 (Theorem 3):
  - Second-order asymptotic, non-vanishing vs first-order asymptotic, vanishing
  - Any measurement dependent channel vs a measurement dependent BSC
  - Multidimensional vs one-dimensional

Further Remarks

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    - Second-order asymptotic, non-vanishing vs first-order asymptotic, vanishing
    - Any measurement dependent channel vs a measurement dependent BSC
    - Multidimensional vs one-dimensional
- Consistent with intuition that searching for a moving d-dimensional target is roughly equivalent to searching for a $2d$-dimensional target
    - Analysis is totally different: account for all trajectories and twist of location and velocity
    - Much more complicated: time complexity is $O(n^{2d+1} v_+^d M^{2d})$ to search for a moving target v.s. $O(nM^{2d})$ to search for a stationary target when the target resolution is $\frac{1}{M}$

An Important Implication: Phase Transition

- Minimal excess-resolution probability $\varepsilon^*(n, d, \delta)$ when $nv_+ = o(\sqrt{n})$

$$\varepsilon^*(n, d, \delta) = \Phi\left(\frac{-d\log\delta - nC}{\sqrt{nV_\varepsilon}}\right) + o(1)$$

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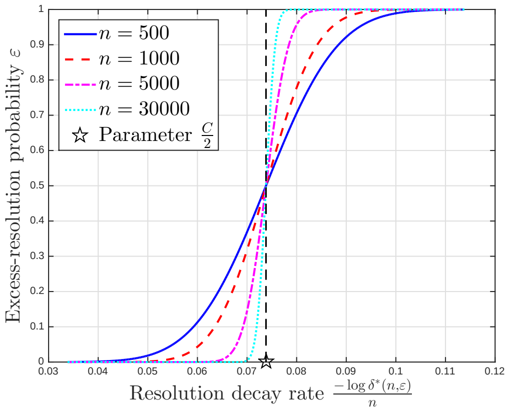
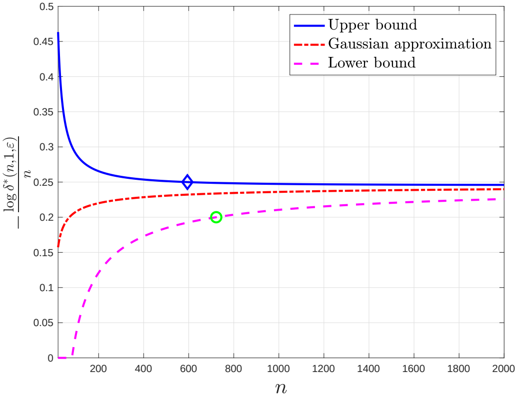
- BSC with crossover probability $(2|\mathcal{A}| + 0.5) \times 0.2$

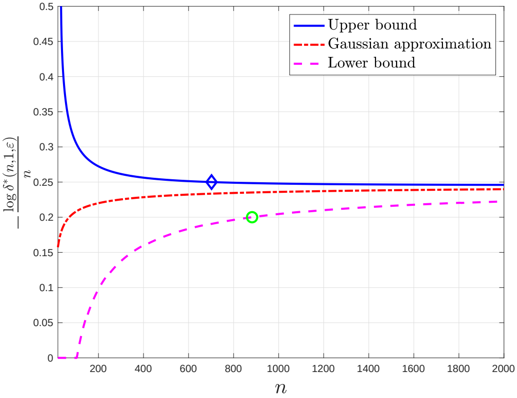


The Impact of the Maximal Speed for A One-Dimensional Target

- Consider uniformly distributed location $S \in [0, 1]$ and velocity $V \in [-v_+, v_+]$
- Consider BSC with crossover probability $(|\mathcal{A}| + 0.5) \times 0.05$ and set $\varepsilon = 0.1$.
- Gaussian approximation (Second-order asymptotic approximation)



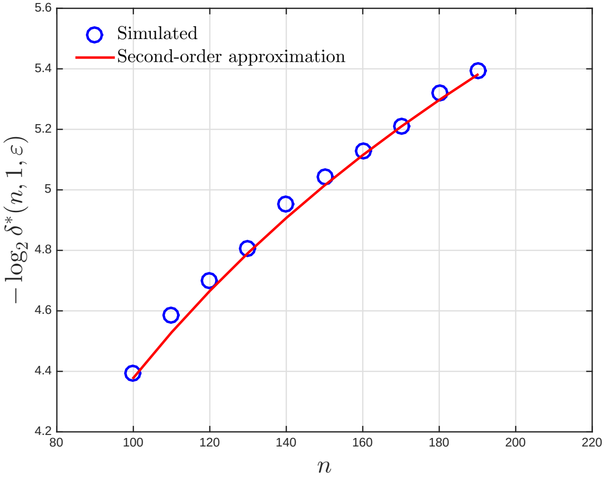
(a) $v_+ = \frac{1}{n}$



(b) $v_+ = \frac{\log n}{n}$

Numerical Simulation

- $nv_+ = 0.1$
- 10^4 independent trials for each n



Reflections and Future Research Directions

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