Resolution Limits of 20 Questions Search Strategies for Moving Targets

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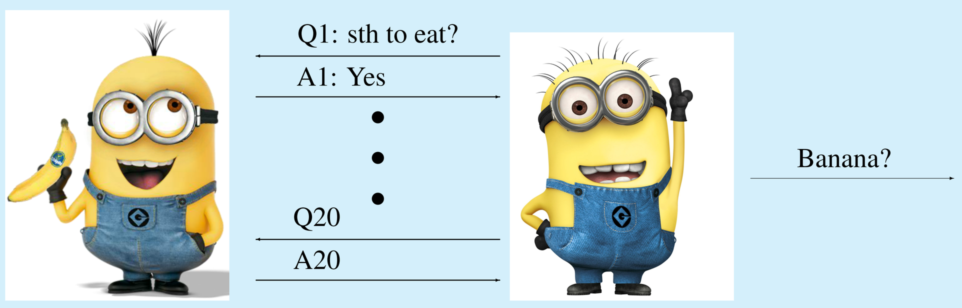
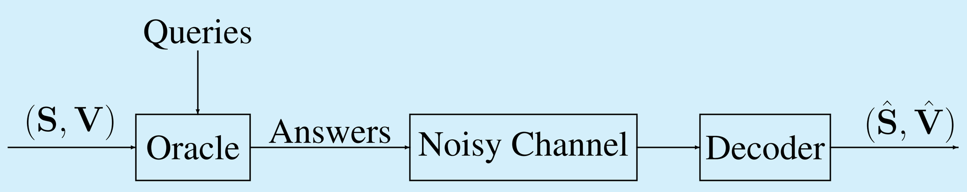
Origin: 20 Questions Game



- Two players: oracle and questioner
- Querying and answering game with a budget (cf. [1]-[3])
- Noiseless: oracle is always honest
- Key: design smart queries to extract the secret as soon as possible

Problem Formulation



- Target: accurately estimate the trajectory of a moving target with unknown initial location $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{S}^d$ and moving velocity $\mathbf{V} = (V_1, \ldots, V_d) \in \mathcal{V}^d$
- $\mathcal{S} = [0, 1]$, $\mathcal{V} = [-v_+, v_+]$ and $(\mathbf{S}, \mathbf{V}) \sim f_{\mathbf{SV}}$.
- Task: design queries and decoder (scheme/strategy/policy)
- Motivation: applications such as localizing a moving target using a sensor network

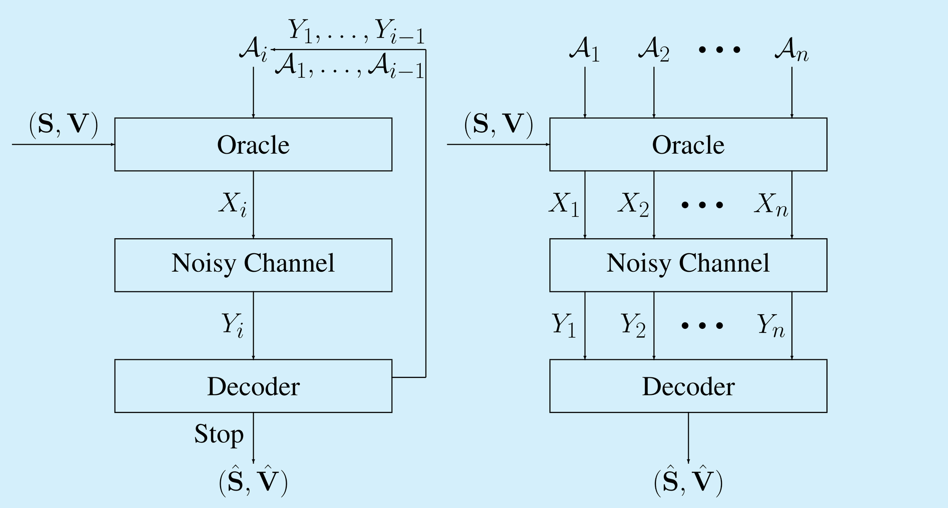
The Torus Model

- Consider a target with initial location $\mathbf{s} = (s_1, \ldots, s_d) \in [0, 1]^d$ and velocity $\mathbf{v} = (v_1, \ldots, v_d) \in \mathcal{V}^d$
- Torus model: at each time point $t \in [n]$ and in each dimension $i \in [d]$, the location of the target is

$$\mathrm{l}(s_i, v_i, t) := \begin{cases} 1 & \text{if } \mathrm{mod}(s_i + tv_i, 1) = 0, \\ s_i + tv_i - \lfloor s_i + tv_i \rfloor & \text{if } s_i + tv_i \in \bigcup_{h \in \mathbb{N}} [2h, 2h+1), \\ \lceil s_i + tv_i \rceil - (s_i + tv_i) & \text{otherwise,} \end{cases}$$

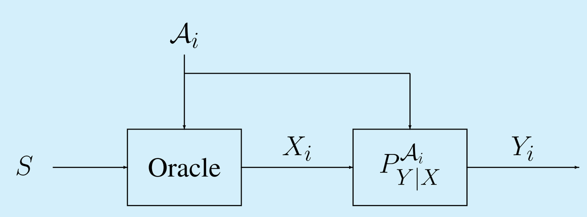
  - the target moves within a unit length interval
  - similar to the unit circle model studied by Kaspi, Shayevitz and Javadi (T-IT 2018)
- The location of the target, denoted as $\mathrm{l}(\mathbf{s}, \mathbf{v}, t)$, is a d-dimensional vector $(\mathrm{l}(s_1, v_1, t), \ldots, \mathrm{l}(s_d, v_d, t))$. At each dimension,

Adaptive and Non-adaptive Querying



- A query $\mathcal{A}_i$ asks whether the target lies in a certain region $\mathcal{A}_i \subseteq [0, 1]^d$ at a discrete time $i \in [n]$ and X_i is the noiseless answer
- *Non-adaptive querying*
  - queries are designed simultaneously offline
  - lower cost, faster execution time due to parallelizability

Measurement Dependent Noisy Channel



- Binary Input alphabet $\mathcal{X}$ and arbitrary output alphabet $\mathcal{Y}$
- For any $\mathcal{A} \subseteq [0, 1]^d$, $P^{\mathcal{A}}_{Y|X}$ depends on $\mathcal{A}$ through its size via a bounded Lipschitz continuous function $f : [0, 1] \to \mathbb{R}_+$
- Example: for any parameter $\zeta \in (0, 1)$ and any $\mathcal{A} \subseteq [0, 1]$
  - Measurement dependent binary symmetric channel (BSC)

$P^{\mathcal{A}}_{Y\|X}$	$X = 0$	$X = 1$
$Y = 0$	$1 - \zeta f(\|\mathcal{A}\|)$	$\zeta f(\|\mathcal{A}\|)$
$Y = 1$	$\zeta f(\|\mathcal{A}\|)$	$1 - \zeta f(\|\mathcal{A}\|)$

Definition of Query Procedure

Given any $n \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and $\varepsilon \in [0, 1]$, to search for a moving target over a d-dimensional torus, an $(n, d, \delta, \varepsilon)$-non-adaptive query procedure consists of

- n queries $\mathcal{A}^n$, where for each $i \in [n]$, we query whether the target locates inside a Lebesgue measurable subset $\mathcal{A}_i \in [0, 1]^d$
- and a decoder $g : \mathcal{Y}^n \to [0, 1]^d \times \mathcal{V}^d$

such that the excess-resolution probability satisfies

$$\mathrm{P_e}(n, d, \delta) := \sup_{f_{\mathbf{SV}}} \Pr\left\{ \max_{t \in [0:n]} \|l(\hat{\mathbf{S}}, \hat{\mathbf{V}}, t) - l(\mathbf{S}, \mathbf{V}, t)\|_\infty > \delta \right\} \leq \varepsilon,$$

Accurate estimation of the trajectory implies accurate estimate of the initial location and velocity, and vice versa

- $|\hat{S}_i - S_i| < \frac{\delta}{2}$ and $n|\hat{V}_i - V_i| < \frac{\delta}{2}$ implies accurate estimation of the trajectory, i.e., $\max_{t \in [0:n]} \|l(\hat{\mathbf{S}}, \hat{\mathbf{V}}, t) - l(\mathbf{S}, \mathbf{V}, t)\|_\infty \leq \delta$
- $|\hat{S}_i - S_i| > \delta$ or $n|\hat{V}_i - V_i| > 2\delta$ implies poor estimate

Fundamental Limit

- Given any number of queries $n \in \mathbb{N}$ and $\varepsilon \in [0, 1]$,

$$\delta^*(n, d, \varepsilon) := \inf\{\delta \in \mathbb{R}_+ : \exists \text{ an } (n, d, \delta, \varepsilon)\text{−non−adaptive query procedure}\}.$$

  - minimal non-asymptotic resolution achievable by any non-adaptive query procedure with n queries and excess-resolution probability ε
- Dual quantity (sample complexity):

$$n^*(d, \delta, \varepsilon) = \inf\{n \in \mathbb{N} : \delta^*(n, d, \varepsilon) \leq \delta\}$$

Preliminaries

- $P_X = \mathrm{Bern}(p)$ denotes the Bernoulli distribution
- $P^q_{Y|X}$ denotes $P^{\mathcal{A}}_{Y|X}$ when $f(|\mathcal{A}|) = q$
- $P^{p,q}_Y$ denotes the distribution on $\mathcal{Y}$ induced by P_X and $P^q_{Y|X}$
- For any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, define the mutual information density

$$\imath_{p,q}(x; y) := \log P^q_{Y|X}(y|x) - \log P^{p,q}_Y(y).$$

- "Capacity" of measurement dependent channels $\{P^q_{Y|X}\}_{q \in [0,1]}$

$$C := \max_{p \in [0,1]} \mathbb{E}[\imath_{p,f(p)}(X; Y)], (X, Y) \sim \mathrm{Bern}(p) \times P^{f(p)}_{Y|X}$$

- "Dispersion" of measurement dependent channels

$$\mathrm{V}_\varepsilon := \begin{cases} \min_{p \in \mathcal{P}_{\mathrm{ca}}} \mathrm{Var}[\imath_{p,f(p)}(X; Y)] & \text{if } \varepsilon < 0.5, \\ \max_{p \in \mathcal{P}_{\mathrm{ca}}} \mathrm{Var}[\imath_{p,f(p)}(X; Y)] & \text{if } \varepsilon > 0.5. \end{cases}$$

Main Result

Theorem 1 *For any $\varepsilon \in (0, 1)$ and finite $d \in \mathbb{N}$, the minimal achievable resolution $\delta^*(n, d, \varepsilon)$ satisfies the following properties*

- *if $nv_+ = O(n^t)$ for $t \in [0.5, 1)$,*

$$-2d \log \delta^*(n, d, \varepsilon) = nC + O(nv_+);$$

- *if $nv_+ = O(n^t)$ for $t \in [0, 0.5)$*

$$-2d \log \delta^*(n, d, \varepsilon) = nC + \sqrt{n\mathrm{V}_\varepsilon}\Phi^{-1}(\varepsilon) + O(\max\{nv_+, \log n\});$$

Discussions

- Theorem 1 is tight under maximal speed constraint v_+
- Refines the result by Kaspi *et al.*, TIT 2018 (Theorem 3):
  - Non-asymptotic, non-vanishing vs asymptotic, vanishing
  - Any measurement dependent channel vs a measurement dependent BSC
  - Multidimensional vs one-dimensional
- Strong converse holds

$$\lim_{n \to \infty} -\frac{1}{n} \log \delta^*(n, d, \varepsilon) = \frac{C}{2d}.$$

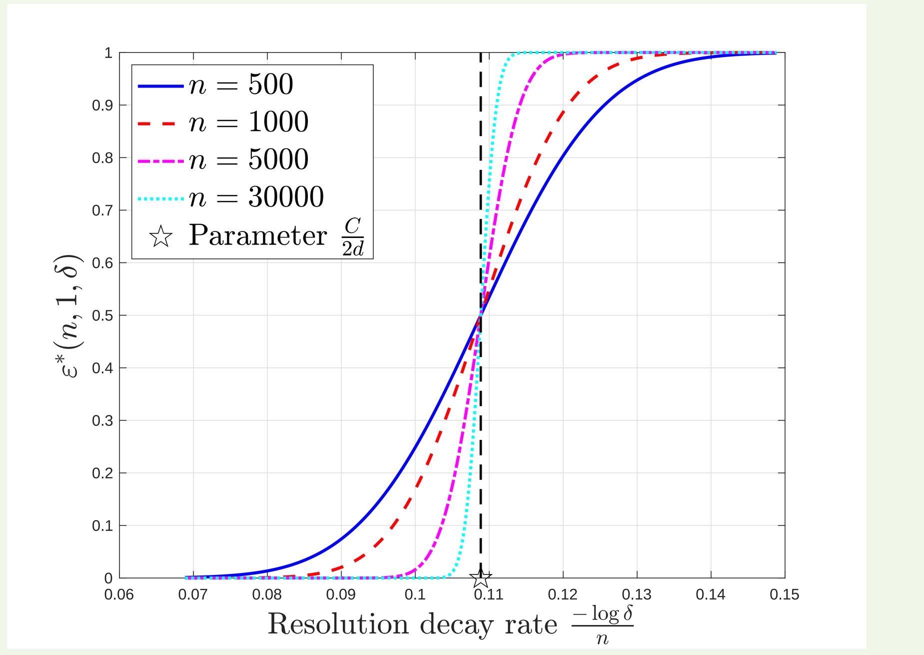
- Proof ideas: finite blocklength channel coding + analysis of the number of quantized trajectories

Phase Transition

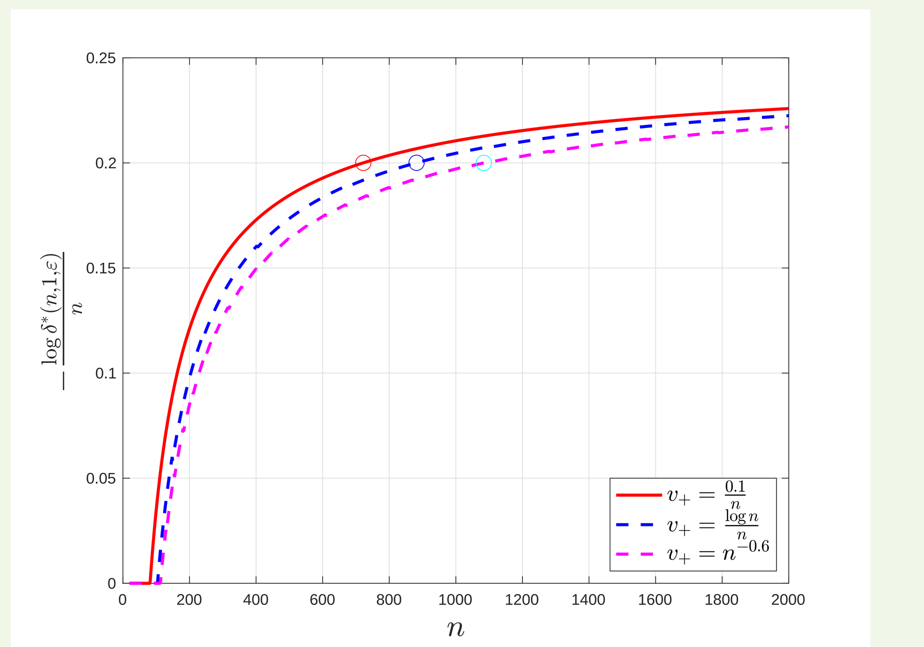
- Minimal excess-resolution probability $\varepsilon^*(n, d, \delta)$
- For maximal speed such that $nv_+ = o(\sqrt{n})$

$$\varepsilon^*(n, d, \delta) = \Phi\left(\frac{-d \log \delta - nC}{\sqrt{n\mathrm{V}_\varepsilon}}\right) + o(1)$$

- Numerical illustration for a measurement dependent BSC with parameter $\zeta = 0.2$ and the Lipschitz continuous function $f(q) = q + 0.5$ when $d = 1$

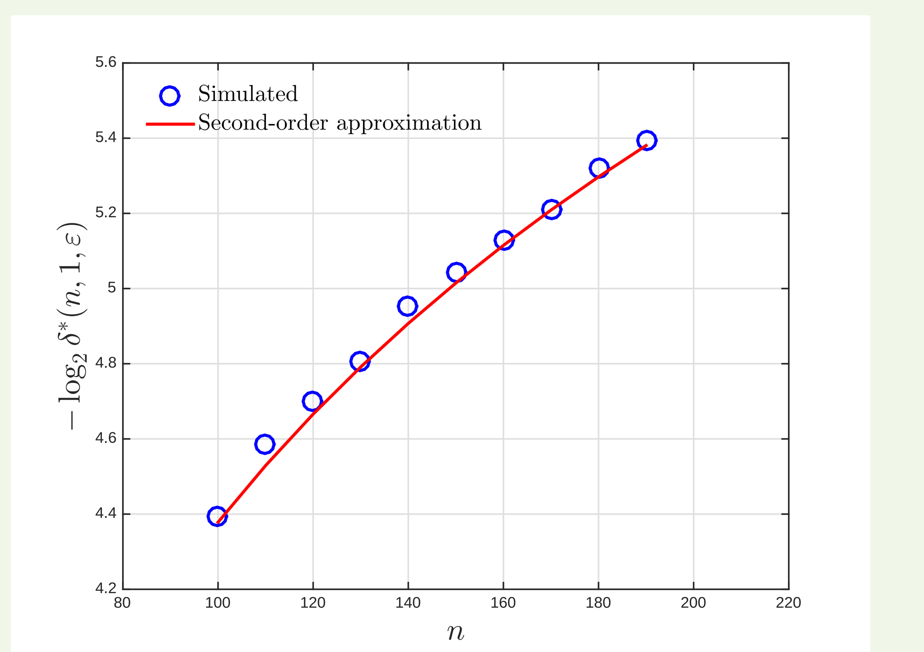


Impact of Maximal Speed



- Measurement dependent BSC with $\zeta = 0.05$ and $f(q) = 0.5 + q$
- The larger the maximal speed, the harder to search

Simulation of Proposed Query Procedure



- Maximal speed $v_+ = \frac{0.1}{n}$

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